

Yokonuma-Schur algebras

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Abstract

In this paper, we define the Yokonuma-Schur algebra $YS_q(r, n)$ as the endomorphism algebra of a permutation module for the Yokonuma-Hecke algebra $Y_{r,n}(q)$. We prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis following the approach in [DJM], and we further show that it is a quasi-hereditary cover of $Y_{r,n}(q)$ in the sense of Rouquier following [HM2]. We also introduce the tilting modules for $YS_q(r, n)$. In the appendix, we define and study the cyclotomic Yokonuma-Schur algebra in a similar way.

Keywords: Yokonuma-Hecke algebras; Yokonuma-Schur algebras; Cellular algebras; Tilting modules; Cyclotomic Yokonuma-Schur algebras

1 Introduction

The Yokonuma-Hecke algebra was first introduced by Yokonuma [Yo] as a centralizer algebra associated to the permutation representation of a Chevalley group G with respect to a maximal unipotent subgroup of G . Juyumaya [Ju1] gave a new presentation of the Yokonuma-Hecke algebra, which is commonly used for studying this algebra.

The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is a quotient of the group algebra of the modular framed braid group $(\mathbb{Z}/r\mathbb{Z}) \wr B_n$, where B_n is the braid group of type A on n strands. It can also be regraded as a deformation of the group algebra of the complex reflection group $G(r, 1, n)$, which is isomorphic to the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, where \mathfrak{S}_n is the symmetric group on n letters. It is well-known that there exists another deformation of the group algebra of $G(r, 1, n)$, the Ariki-Koike algebra $H_{r,n}$ [AK]. The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is quite different from $H_{r,n}$. For example, the Iwahori-Hecke algebra of type A is canonically a subalgebra of $H_{r,n}$, whereas it is an obvious quotient of $Y_{r,n}(q)$, but not an obvious subalgebra of it.

Recently, many people are largely motivated to study $Y_{r,n}(q)$ in order to construct its associated knot invariant; see the papers [Ju2], [JuL] and [ChL]. In particular, Juyumaya [Ju2] found a basis of $Y_{r,n}(q)$, and then defined a Markov trace on it.

Some other people are particularly interested in the representation theory of $Y_{r,n}(q)$, and also its application to knot theory. Chlouveraki and Poulain d'Andecy [ChP1] gave explicit formulas for all irreducible representations of $Y_{r,n}(q)$ over $\mathbb{C}(q)$, and obtained a semisimplicity criterion for it. In their subsequent paper [ChP2], they defined

and studied the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$, and constructed several bases for them, and then showed how to define Markov traces on these algebras. Moreover, they gave the classification of irreducible representations of $Y_{r,n}^d(q)$ in the generic semisimple case, defined the canonical symmetrizing form on it and computed the associated Schur elements directly.

Recently, Jacon and Poulain d'Andecy [JP] constructed an explicit algebraic isomorphism between the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and a direct sum of matrix algebras over tensor products of Iwahori-Hecke algebras of type A , which is in fact a special case of the results by G. Lusztig [Lu, Section 34]. This allows them to give a description of the modular representation theory of $Y_{r,n}(q)$ and a complete classification of all Markov traces for it. Chlouveraki and Sécherre [ChS, Theorem 4.3] proved that the affine Yokonuma-Hecke algebra is a particular case of the pro- p -Iwahori-Hecke algebra defined by Vignéras in [Vi].

Espinoza and Ryom-Hansen [ER] gave a new proof of Jacon and Poulain d'Andecy's isomorphism theorem by giving a concrete isomorphism between $Y_{r,n}(q)$ and Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. Moreover, they showed that $Y_{r,n}(q)$ is a cellular algebra by giving an explicit cellular basis. Combining the results of [DJM] with those of [ER], we [C1] proved that the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$ is cellular by constructing an explicit cellular basis, and showed that the Jucys-Murphy elements for $Y_{r,n}^d(q)$ are JM-elements in the abstract sense introduced by Mathas [Ma3].

We [CW] have established an equivalence between a module category of the affine (resp. cyclotomic) Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ (resp. $Y_{r,n}^d(q)$) and its suitable counterpart for a direct sum of tensor products of affine Hecke algebras of type A (resp. cyclotomic Hecke algebras), which allows us to give the classification of simple modules of affine Yokonuma-Hecke algebras and of the associated cyclotomic Yokonuma-Hecke algebras over an algebraically closed field of characteristic $p = 0$ or $(p, r) = 1$, and also describe the classification of blocks for these algebras. In addition, the modular branching rules for cyclotomic (resp. affine) Yokonuma-Hecke algebras are obtained, and they are further identified with crystal graphs of integrable modules for affine lie algebras of type A . In a subsequent paper, we [C2] have established an explicit algebra isomorphism between the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and a direct sum of matrix algebras over tensor products of affine Hecke algebras of type A . As an application, we proved that $\widehat{Y}_{r,n}(q)$ is affine cellular in the sense of Koenig and Xi, and studied its homological properties.

In [DJM], they constructed a cellular basis for the cyclotomic q -Schur algebra $\mathcal{S}(\Lambda)$ by firstly constructing a cellular basis for the Ariki-Koike algebra $H_{r,n}$. They further obtained a complete set of non-isomorphic irreducible $\mathcal{S}(\Lambda)$ -modules and showed that it is quasi-hereditary. Now, there exists a cellular basis on $Y_{r,n}(q)$ by [ER], it is natural to try to define and study the corresponding Schur algebra for the Yokonuma-Hecke algebra $Y_{r,n}(q)$ by using this cellular basis.

In this paper, we will define the Yokonuma-Schur algebra $YS_q(r, n)$ as the endomorphism algebra of a permutation module associated to the Yokonuma-Hecke algebra $Y_{r,n}(q)$. Combining the results of [DJM] with those of [SS], we prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis, and further prove that it is quasi-hereditary. We also investigate the indecomposable tilting modules for $YS_q(r, n)$ and prove that they are self-dual.

This paper is organized as follows. In Section 2, we recall the definition of the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and the construction of a cellular basis of $Y_{r,n}(q)$ following [ER]. In Section 3, we will define the Yokonuma-Schur algebra $YS_q(r, n)$ as the endomorphism algebra of a permutation module associated to the Yokonuma-Hecke algebra $Y_{r,n}(q)$. We prove that $YS_q(r, n)$ is cellular by constructing an explicit cellular basis, and further prove that it is quasi-hereditary by combining the results of [DJM] with those of [SS]. In Section 4, following the approach in [HM2], we will construct an exact functor from the category of $YS_q(r, n)$ -modules to the category of $Y_{r,n}(q)$ -modules. In Section 5, we introduce the tilting modules for $YS_q(r, n)$ and the closely related Young modules for $Y_{r,n}(q)$ following [Ma2]. In the appendix, we will generalize these results to define and study the cyclotomic Yokonuma-Schur algebra by using the cellular basis of $Y_{r,n}^d(q)$ constructed in [C1]. Since this approach is very similar, we only mention the main results and skip all the details.

Many ideas of this paper originate from the references [DJM, Ma2, SS], although it should be noted that the basic set-up here is different from theirs; anyhow, we expect that the Yokonuma-Schur algebra and its cyclotomic analog defined here deserve further study.

2 Cellular Bases for Yokonuma-Hecke algebras

In this section, we recall the definition of the Yokonuma-Hecke algebra $Y_{r,n}(q)$ and the construction of a cellular basis of $Y_{r,n}(q)$ presented in [ER, Section 4].

Let $r, n \in \mathbb{N}$, $r \geq 1$, and let $\zeta = e^{2\pi i/r}$. Let q be an indeterminate and let $\mathcal{R} = \mathbb{Z}[\frac{1}{r}][q, q^{-1}, \zeta]$. The Yokonuma-Hecke algebra $Y_{r,n} = Y_{r,n}(q)$ is an \mathcal{R} -associative algebra generated by the elements $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ satisfying the following relations:

$$g_i g_j = g_j g_i \quad \text{for all } i, j = 1, 2, \dots, n-1 \text{ such that } |i - j| \geq 2,$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for all } i = 1, 2, \dots, n-1,$$

$$t_i t_j = t_j t_i \quad \text{for all } i, j = 1, 2, \dots, n,$$

$$g_i t_j = t_{s_i(j)} g_i \quad \text{for all } i = 1, 2, \dots, n-1 \text{ and } j = 1, 2, \dots, n,$$

$$t_i^r = 1 \quad \text{for all } i = 1, 2, \dots, n.$$

$$g_i^2 = 1 + (q - q^{-1})e_i g_i \quad \text{for all } i = 1, 2, \dots, n-1,$$

where s_i is the transposition $(i, i+1)$, and for each $1 \leq i \leq n-1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}.$$

Note that the elements e_i are idempotents in $Y_{r,n}$. The elements g_i are invertible, with the inverse given by

$$g_i^{-1} = g_i - (q - q^{-1})e_i \quad \text{for all } i = 1, 2, \dots, n-1.$$

Let $w \in \mathfrak{S}_n$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w . By Matsumoto's lemma, the element $g_w := g_{i_1} g_{i_2} \cdots g_{i_r}$ does not depend on the choice of the reduced expression of w , that is, it is well-defined. Let l denote the length function on \mathfrak{S}_n . Then we have

$$g_i g_w = \begin{cases} g_{s_i w} & \text{if } l(s_i w) > l(w); \\ g_{s_i w} + (q - q^{-1})e_i g_w & \text{if } l(s_i w) < l(w). \end{cases}$$

Using the multiplication formulas given above, Juyumaya [Ju2] has proved that the following set is an \mathcal{R} -basis for $Y_{r,n}$:

$$\mathcal{B}_{r,n} = \{t_1^{k_1} \cdots t_n^{k_n} g_w \mid k_1, \dots, k_n \in \mathbb{Z}/r\mathbb{Z}, w \in \mathfrak{S}_n\}.$$

Thus, $Y_{r,n}$ is a free \mathcal{R} -module of rank $r^n n!$.

Let $i, k \in \mathbf{s} = \{1, 2, \dots, n\}$ and set

$$e_{i,k} := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_k^{-s}.$$

Note that $e_{i,i} = 1$, $e_{i,k} = e_{k,i}$, and that $e_{i,i+1} = e_i$. For any nonempty subset $I \subseteq \mathbf{s}$ we define the following element E_I by

$$E_I = \prod_{i,j \in I; i < j} e_{i,j},$$

where by convention $E_I = 1$ if $|I| = 1$.

We also need a further generalization of this. We say that the set $A = \{I_1, I_2, \dots, I_k\}$ is a set partition of \mathbf{s} if the I_j 's are nonempty and disjoint subsets of \mathbf{s} , and their union is \mathbf{s} . We refer to them as the blocks of A . We denote by \mathcal{SP}_n the set of all set partitions of \mathbf{s} . For $A = \{I_1, I_2, \dots, I_k\} \in \mathcal{SP}_n$ we then define $E_A = \prod_j E_{I_j}$.

$\mu = (\mu_1, \dots, \mu_k)$ is called a composition of n if it is a finite sequence of nonnegative integers whose sum is n . A composition μ is a partition of n if its parts are non-increasing. We write $\mu \models n$ (resp. $\lambda \vdash n$) if μ is a composition (resp. partition) of n , and we define $|\mu| := n$ (resp. $|\lambda| := n$). An r -composition (resp. r -partition) of n is an ordered r -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ of compositions (resp. partitions) $\lambda^{(k)}$ such

that $\sum_{k=1}^r |\lambda^{(k)}| = n$. We denote by $\mathcal{C}_{r,n}$ (resp. $\mathcal{P}_{r,n}$) the set of r -compositions (resp. r -partitions) of n .

We associate a Young diagram to a composition μ , which is the set

$$[\mu] = \{(i, j) \mid i \geq 1 \text{ and } 1 \leq j \leq \mu_i\}.$$

We will regard $[\mu]$ as an array of boxes, or nodes, in the plane. For $\mu \models n$, we define a μ -tableau by replacing each node of $[\mu]$ by one of the integers $1, 2, \dots, n$, allowing no repeats. For $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \mathcal{C}_{r,n}$, a μ -tableau $\mathbf{t} = (\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(r)})$ is a bijection $\mathbf{t} : [\mu] \rightarrow \{1, 2, \dots, n\}$, where $[\mu] = ([\mu^{(1)}], \dots, [\mu^{(r)}])$ is the diagram of μ and each $\mathbf{t}^{(i)}$ is a $\mu^{(i)}$ -tableau.

For $\mu \models n$, we say that a μ -tableau \mathbf{t} is row standard if the entries in each row of \mathbf{t} increase from left to right. A μ -tableau \mathbf{t} is standard if μ is a partition, \mathbf{t} is row standard and the entries in each column increase from top to bottom. For $\mu \in \mathcal{C}_{r,n}$ (resp. $\mu \in \mathcal{P}_{r,n}$), a μ -tableau \mathbf{t} is called row standard (resp. standard) if all $\mathbf{t}^{(i)}$ are row standard (resp. standard). If $\mu \in \mathcal{C}_{r,n}$, we denote by $\text{r-Std}(\mu)$ the set of row standard μ -tableaux. For $\lambda \in \mathcal{P}_{r,n}$, let $\text{Std}(\lambda)$ denote the set of standard λ -tableaux. For each $\mu \in \mathcal{C}_{r,n}$, we denote by \mathbf{t}^μ the μ -tableau in which $1, 2, \dots, n$ appear in increasing order from left to right along the rows of the first component of $[\mu]$, and then along the rows of the second component, and so on.

For each $\mu \in \mathcal{C}_{r,n}$ and a row standard μ -tableau \mathbf{s} , let $d(\mathbf{s})$ be the unique element of \mathfrak{S}_n such that $\mathbf{s} = \mathbf{t}^\mu d(\mathbf{s})$. Then $d(\mathbf{s})$ is a distinguished right coset representative of \mathfrak{S}_μ in \mathfrak{S}_n , that is, $l(wd(\mathbf{s})) = l(w) + l(d(\mathbf{s}))$ for any $w \in \mathfrak{S}_\mu$. In this way, we obtain a bijection between the set $\text{r-Std}(\mu)$ of row standard μ -tableaux and the set \mathcal{D}_μ of distinguished right coset representatives of \mathfrak{S}_μ in \mathfrak{S}_n .

Let $\mu \in \mathcal{C}_{r,n}$ and \mathbf{t} be a μ -tableau. For $j = 1, \dots, n$, we define $p_{\mathbf{t}}(j) = k$ if j appears in the k -th component $\mathbf{t}^{(k)}$ of \mathbf{t} . When $\mathbf{t} = \mathbf{t}^\mu$, we write $p_\mu(j)$ instead of $p_{\mathbf{t}^\mu}(j)$.

We now fix once and for all a total order on the set of r -th roots of unity via $\{\zeta^k \mid k = 0, 1, \dots, r-1\} = \{\zeta_1, \zeta_2, \dots, \zeta_r\}$. Then we define a set partition $A_\lambda \in \mathcal{SP}_n$ for any r -composition λ .

Definition 2.1. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{C}_{r,n}$. Suppose that we choose all $1 \leq i_1 < i_2 < \dots < i_p \leq r$ such that $\lambda^{(i_1)}, \lambda^{(i_2)}, \dots, \lambda^{(i_p)}$ are the nonempty components of λ . Define $a_k := \sum_{j=1}^k |\lambda^{(i_j)}|$ for $1 \leq k \leq p$. Then the set partition A_λ associated with λ is defined as

$$A_\lambda := \{\{1, \dots, a_1\}, \{a_1 + 1, \dots, a_2\}, \dots, \{a_{p-1} + 1, \dots, n\}\},$$

which may be written as $A_\lambda = \{I_1, I_2, \dots, I_p\}$, and be referred to the blocks of A_λ in the order given above.

Definition 2.2. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{C}_{r,n}$, and let $a_k := \sum_{j=1}^k |\lambda^{(i_j)}|$ ($1 \leq k \leq p$) be as above. Then we define

$$u_\lambda := u_{a_1, i_1} u_{a_2, i_2} \cdots u_{a_p, i_p},$$

where $u_{i,k} = \prod_{l=1; l \neq k}^r (t_i - \zeta_l)$ for $1 \leq i \leq n$ and $1 \leq k \leq r$.

Definition 2.3. Let $\lambda \in \mathcal{C}_{r,n}$. We set $U_\lambda := u_\lambda E_{A_\lambda}$, and define $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} g_w$. Then we define the element m_λ of $Y_{r,n}$ as follows:

$$m_\lambda := U_\lambda x_\lambda = u_\lambda E_{A_\lambda} x_\lambda.$$

Let $*$ denote the \mathcal{R} -linear anti-automorphism of $Y_{r,n}$, which is determined by $g_i^* = g_i$ and $t_j^* = t_j$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Definition 2.4. Let $\lambda \in \mathcal{C}_{r,n}$, and let \mathfrak{s} and \mathfrak{t} be row standard λ -multitableaux. We then define $m_{\mathfrak{s}\mathfrak{t}} = g_{d(\mathfrak{s})}^* m_\lambda g_{d(\mathfrak{t})}$.

For each $\mu \in \mathcal{P}_{r,n}$, let $Y_{r,n}^{\triangleright \mu}$ be the \mathcal{R} -submodule of $Y_{r,n}$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in \text{Std}(\lambda)$ for various $\lambda \in \mathcal{P}_{r,n}$ such that $\lambda \triangleright \mu$.

Theorem 2.5. (See [ER, Theorem 20].) *The algebra $Y_{r,n}$ is a free \mathcal{R} -module with a cellular basis*

$$\mathcal{B}_{r,n} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } r\text{-partition } \lambda \text{ of } n\},$$

that is, the following properties hold.

(i) The \mathcal{R} -linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$ ($m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{B}_{r,n}$) is an anti-automorphism on $Y_{r,n}$.

(ii) For a given $h \in Y_{r,n}$ and $\mathfrak{t} \in \text{Std}(\mu)$, there exist $r_{\mathfrak{v}} \in \mathcal{R}$ such that for all $\mathfrak{s} \in \text{Std}(\mu)$, we have

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \text{Std}(\mu)} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \quad \text{mod } Y_{r,n}^{\triangleright \mu},$$

where $r_{\mathfrak{v}}$ may depend on $\mathfrak{v}, \mathfrak{t}$ and h , but not on \mathfrak{s} .

For each $\lambda \in \mathcal{P}_{r,n}$, let \overline{m}_λ be the image of m_λ under the algebra homomorphism $Y_{r,n} \rightarrow Y_{r,n}/Y_{r,n}^{\triangleright \lambda}$. We denote by S^λ the right $Y_{r,n}$ -submodule of $Y_{r,n}/Y_{r,n}^{\triangleright \lambda}$ generated by \overline{m}_λ , which is called the Specht module associated to λ . By Theorem 2.5, S^λ is a free \mathcal{R} -module with basis $\{\overline{m}_\lambda g_{d(\mathfrak{t})} \mid \mathfrak{t} \in \text{Std}(\lambda)\}$. We can define an associative symmetric bilinear form on S^λ by

$$m_\lambda g_{d(\mathfrak{s})} g_{d(\mathfrak{t})}^* m_\lambda \equiv \langle \overline{m}_\lambda g_{d(\mathfrak{s})}, \overline{m}_\lambda g_{d(\mathfrak{t})} \rangle m_\lambda \quad \text{mod } Y_{r,n}^{\triangleright \lambda}.$$

Let $\text{rad} S^\lambda = \{u \in S^\lambda \mid \langle u, v \rangle = 0 \text{ for all } v \in S^\lambda\}$. Consequently, $\text{rad} S^\lambda$ is a $Y_{r,n}$ -submodule of S^λ . Let $D^\lambda = S^\lambda / \text{rad} S^\lambda$ for each $\lambda \in \mathcal{P}_{r,n}$. By a general theory of cellular basis, if $\mathcal{R} = \mathbb{K}$ is a field, the set $\{D^\lambda \neq 0 \mid \lambda \in \mathcal{P}_{r,n}\}$ gives a complete set of non-isomorphic irreducible $Y_{r,n}$ -modules. In fact, by [ER, Theorem 7 and (46)] (see also [JP, §4.1] and [CW, Theorem 6.3]), $\{\lambda \in \mathcal{P}_{r,n} \mid D^\lambda \neq 0\}$ is just the set $\mathcal{K}_{r,n}$, where

$$\mathcal{K}_{r,n} = \{\lambda \in \mathcal{P}_{r,n} \mid \lambda = (\lambda_1, \dots, \lambda_r) \text{ with each } \lambda_i \text{ being an } e\text{-restricted partition}\}.$$

3 Yokonuma-Schur algebra and its cellular basis

For an r -composition λ of n , a λ -tableau $S = (S^{(1)}, \dots, S^{(r)})$ is a map $S : [\lambda] \rightarrow \{1, \dots, n\} \times \{1, \dots, r\}$, which can be regarded as the diagram $[\lambda]$, together with an ordered pair (i, k) ($1 \leq i \leq n, 1 \leq k \leq r$) attached to each node. Given $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$, a λ -tableau S is said to be of type μ if the number of (i, k) in the entry of S is equal to $\mu_i^{(k)}$. Given $\mathfrak{s} \in \text{Std}(\lambda)$, $\mu(\mathfrak{s})$, a λ -tableau of type μ , is defined by replacing each entry m in \mathfrak{s} by (i, k) if m is in the i -th row of the k -th component of \mathfrak{t}^μ .

We define a total order on the set of pairs (i, k) by $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$. Let $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$. Suppose that $S = (S^{(1)}, \dots, S^{(r)})$ is a λ -tableau of type μ . S is said to be semistandard if each component $S^{(k)}$ is non-decreasing in rows, strictly increasing in columns, and all entries of $S^{(k)}$ are of the form (i, l) with $l \geq k$. We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard λ -tableaux of type μ .

Let us consider the special case when $r = 1$. Suppose that $\lambda, \mu \in \mathcal{C}_{1,n}$. A λ -tableau S of type μ is said to be row semistandard if the entries in each row of S are non-decreasing. S is said to be semistandard if $\lambda \in \mathcal{P}_{1,n}$, S is row semistandard and the entries in each column are strictly increasing. Assume that $\lambda, \mu \in \mathcal{C}_{1,n}$ and put $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$. Then $\mathcal{D}_{\lambda\mu}$ is the set of minimal length elements in the double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu$, and the map $d \mapsto \mu(\mathfrak{t}^\lambda d)$ gives a bijection between the set $\mathcal{D}_{\lambda\mu}$ and the set of row semistandard λ -tableaux of type μ .

Let $d \in \mathcal{D}_{\lambda\mu}$, and put $S = \mu(\mathfrak{t}^\lambda d)$, $T = \lambda(\mathfrak{t}^\mu d^{-1})$. Then S and T are both row semistandard, and we have

$$\sum_{\substack{y \in \mathcal{D}_\mu \\ \lambda(\mathfrak{t}^\mu y) = T}} q^{l(y)} g_y^* x_\mu = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} q^{l(w)} g_w = \sum_{\substack{x \in \mathcal{D}_\lambda \\ \mu(\mathfrak{t}^\lambda x) = S}} q^{l(x)} x_\lambda g_x. \quad (3.1)$$

For any $\kappa \in \mathcal{C}_{r,n}$, we define its type $\alpha(\kappa)$ by $\alpha(\kappa) = (n_1, \dots, n_r)$ with $n_i = |\kappa^{(i)}|$. Assume that $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$. We define a subset $\mathcal{T}_0^+(\lambda, \mu)$ of $\mathcal{T}_0(\lambda, \mu)$ by

$$\mathcal{T}_0^+(\lambda, \mu) = \{S \in \mathcal{T}_0(\lambda, \mu) \mid \alpha(\lambda) = \alpha(\mu)\}.$$

Take $S \in \mathcal{T}_0(\lambda, \mu)$. One can check that $S \in \mathcal{T}_0^+(\lambda, \mu)$ if and only if each entry of $S^{(k)}$ is of the form (i, k) for some i . Moreover, if $\mathfrak{s} \in \text{Std}(\lambda)$ is such that $\mu(\mathfrak{s}) = S$ with $S \in \mathcal{T}_0^+(\lambda, \mu)$, then the entries of the i -th component of \mathfrak{s} consist of numbers $a_i + 1, \dots, a_{i+1}$, where $a_i = \sum_{k=1}^{i-1} n_k$. In particular, $d(\mathfrak{s}) \in \mathfrak{S}_\alpha$ for $\alpha = \alpha(\lambda)$.

Take $S \in \mathcal{T}_0^+(\lambda, \mu)$. Let $\mathfrak{s}_1 = \text{first}(S)$, which is the unique element of $\text{Std}(\lambda)$ satisfying the property that $\mu(\mathfrak{s}_1) = S$ and that $\mathfrak{s}_1 \supseteq \mathfrak{s}$ for any $\mathfrak{s} \in \text{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$. Let $\alpha = \alpha(\lambda) = \alpha(\mu)$. Then $d = d(\mathfrak{s}_1) \in \mathfrak{S}_\alpha$, which is given as $d = (d_1, \dots, d_r)$ with d_k a distinguished double coset representative in $\mathfrak{S}_{\lambda^{(k)}} \backslash \mathfrak{S}_{n_k} / \mathfrak{S}_{\mu^{(k)}}$. From (3.1) we have

$$\sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s}))} x_\lambda g_{d(\mathfrak{s})} = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} q^{l(w)} g_w = h g_d x_\mu, \quad (3.2)$$

where $h = \sum g_v$, the sum running over certain elements $v \in \mathfrak{S}_\lambda$.

For each $\mu \in \mathcal{C}_{r,n}$, let $M^\mu = m_\mu Y_{r,n}$. The next lemma gives a basis of M^μ as an \mathcal{R} -module.

Lemma 3.1. *For each $\mu \in \mathcal{C}_{r,n}$, $\{m_\mu g_d \mid d \in \mathcal{D}_\mu\}$ is an \mathcal{R} -basis of M^μ .*

Proof. Since $m_\mu g_d = \sum_{w \in \mathfrak{S}_\mu} U_\mu g_{wd}$ for each $d \in \mathcal{D}_\mu$, and hence $\{m_\mu g_d\}$ is linearly independent. Since $m_\mu t_i = \zeta_{p_\mu(i)} m_\mu$ by [ER, Lemma 11(4)] and $m_\mu g_w = q^{l(w)} m_\mu$ for $w \in \mathfrak{S}_\mu$, then the set $\{m_\mu g_d \mid d \in \mathcal{D}_\mu\}$ spans M^μ by the basis theorem. Thus, $\{m_\mu g_d \mid d \in \mathcal{D}_\mu\}$, or equivalently, $\{m_\mu g_{d(\mathfrak{t})} \mid \mathfrak{t} \in \text{r-Std}(\mu)\}$ is an \mathcal{R} -basis of M^μ . \square

We now construct a basis of M^μ related to the cellular basis $\{m_{\mathfrak{st}}\}$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$, we define

$$m_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{st}}.$$

We have

Lemma 3.2. *Let $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$. Then $m_{S\mathfrak{t}} \in M^\mu$.*

Proof. By (3.2) we have

$$\begin{aligned} m_{S\mathfrak{t}} &= \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} g_{d(\mathfrak{s})}^* x_\lambda U_\lambda g_{d(\mathfrak{t})} \\ &= q^{l(d(\mathfrak{t}))} x_\mu g_d^* h^* U_\lambda g_{d(\mathfrak{t})}. \end{aligned}$$

Since $h = \sum g_v$, where $v \in \mathfrak{S}_\lambda$, hence U_λ commutes with h^* by [ER, Lemma 11(3)]. Since $d \in \mathfrak{S}_\alpha$ with $\alpha = \alpha(\lambda)$, U_λ commutes with g_d^* by the same reason. Noting that $\alpha = \alpha(\lambda) = \alpha(\mu)$, we have $U_\lambda = U_\mu$. Thus, we see that

$$x_\mu g_d^* h^* U_\lambda = x_\mu U_\mu g_d^* h^* \in m_\mu Y_{r,n} = M^\mu,$$

and $m_{S\mathfrak{t}} \in M^\mu$ as required. \square

Proposition 3.3. *For each $\mu \in \mathcal{C}_{r,n}$, M^μ is free with an \mathcal{R} -basis*

$$\{m_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$$

Proof. The basis elements $m_{\mathfrak{st}}$ contained in the expression of $m_{S\mathfrak{t}}$ are disjoint for different $m_{S\mathfrak{t}}$. It follows that $m_{S\mathfrak{t}}$ are linearly independent. By Lemma 3.1, M^μ is a free \mathcal{R} -module, and its rank is equal to the number of $\{m_{S\mathfrak{t}}\}$ given in the proposition by [SS, Lemma 2.5(ii) and Corollary 4.5(ii)]. Hence, any element in M^μ can be written as a linear combination of various $m_{S\mathfrak{t}}$ with coefficients in the quotient field of \mathcal{R} . But since the set $\{q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{st}}\}$ is an \mathcal{R} -basis of $Y_{r,n}$, these coefficients are actually in \mathcal{R} . This proves the proposition. \square

Let $\mu, \nu \in \mathcal{C}_{r,n}$ be such that $\alpha(\mu) = \alpha(\nu) = \alpha$. Put $\alpha = (n_1, \dots, n_r)$. Let us take $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha$. We have $d = (d_1, \dots, d_r)$ with $d_k \in \mathcal{D}_{\mu^k \nu^k}$ with respect to \mathfrak{S}_{n_k} . Then we can define a map $\varphi_{\mu\nu}^d : M^\nu \rightarrow M^\mu$ by

$$\varphi_{\mu\nu}^d(m_\nu h) = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} U_\mu g_w h$$

for all $h \in Y_{r,n}$. In fact, by (3.1), we have

$$\sum_{\substack{y \in \mathcal{D}_\nu \cap \mathfrak{S}_\alpha \\ \mu(\mathfrak{t}^\nu y) = T}} q^{l(y)} U_\mu g_y^* x_\nu = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} U_\mu g_w = \sum_{\substack{x \in \mathcal{D}_\mu \cap \mathfrak{S}_\alpha \\ \nu(\mathfrak{t}^\mu x) = S}} q^{l(x)} m_\mu g_x, \quad (3.3)$$

where $S = \mu(\mathfrak{t}^\nu d)$, and $T = \nu(\mathfrak{t}^\mu d^{-1})$ are row semistandard tableaux. Noting that $U_\mu = U_\nu$ and $y \in \mathfrak{S}_\alpha$, we have $U_\mu g_y^* x_\nu = g_y^* U_\nu x_\nu = g_y^* m_\nu$, and $\varphi_{\mu\nu}^d$ is well-defined.

The proof of the next proposition is inspired by that of [SS, Proposition 5.2], although it should be noted that the basic setup there is different from ours. It allows us to restrict ourselves to considering the subset $\mathcal{T}_0^+(\lambda, \mu)$.

Proposition 3.4. *Let $\mu, \nu \in \mathcal{C}_{r,n}$. Then*

- (i) *Assume that $\alpha(\mu) \neq \alpha(\nu)$. Then $\text{Hom}_{Y_{r,n}}(M^\nu, M^\mu) = 0$.*
- (ii) *Assume that $\alpha(\mu) = \alpha(\nu)$. Then $\text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$ is a free \mathcal{R} -module with basis $\{\varphi_{\mu\nu}^d \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$.*

Proof. Suppose that $\varphi \in \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$. Then, for all $h \in Y_{r,n}$, we have $\varphi(m_\nu h) = \varphi(m_\nu)h$; hence φ is completely determined by $\varphi(m_\nu)$. Since $\varphi(m_\nu) \in M^\mu$, by Lemma 3.1, there exist some $c_x \in \mathcal{R}$ such that $\varphi(m_\nu) = \sum_{x \in \mathcal{D}_\mu} c_x m_\mu g_x$. By [ER, Lemma 10(49)], for each k , we have

$$\varphi(m_\nu t_k) = \varphi(\zeta_{p_\nu(k)} m_\nu) = \sum_{x \in \mathcal{D}_\mu} \zeta_{p_\nu(k)} c_x m_\mu g_x. \quad (3.4)$$

Now assume that $c_y \neq 0$ for some $y \in \mathcal{D}_\mu$, which is equal to some $d(\mathfrak{s})$ for some row standard μ -tableau \mathfrak{s} . Then we have

$$(c_y m_\mu g_y) t_k = c_y m_\mu t_{kd(\mathfrak{s})^{-1}} g_y = \zeta_{p_\nu(kd(\mathfrak{s})^{-1})} c_y m_\mu g_y.$$

Since $\mathfrak{s} = \mathfrak{t}^\mu d(\mathfrak{s})$, we have that $p_\nu(kd(\mathfrak{s})^{-1}) = p_\mathfrak{s}(k)$, and hence

$$(c_y m_\mu g_y) t_k = \zeta_{p_\mathfrak{s}(k)} c_y m_\mu g_y. \quad (3.5)$$

By comparing (3.4) and (3.5), we have $p_{\mathfrak{t}^\nu}(k) = p_\mathfrak{s}(k)$ for all $k = 1, \dots, n$. This implies that $\alpha(\mu) = \alpha(\nu)$. Thus, (i) is proved.

Now assume that $\alpha(\mu) = \alpha(\nu) = \alpha$. Since $p_{\mathfrak{t}^\nu}(k) = p_{\mathfrak{t}^\mu}(kd(\mathfrak{s})^{-1})$ for all $k = 1, \dots, n$, we must have $y = d(\mathfrak{s}) \in \mathfrak{S}_\alpha$. Let d be the unique minimal length element in $\mathfrak{S}_\mu y \mathfrak{S}_\nu$. Then $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha$, and a similar argument as in the proof of [Ma1, Theorem 4.8] implies that $c_d \neq 0$. Set $\varphi' = \varphi - c_d \varphi_{\mu\nu}^d$. Then $\varphi' \in \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$, and $\varphi'(m_\nu)$

can be written as $\varphi'(m_\nu) = \sum_{x \in \mathcal{D}_\mu} a_x m_\mu g_x$, where $a_x = c_x$ if $\mathfrak{S}_\mu x \mathfrak{S}_\nu \neq \mathfrak{S}_\mu d \mathfrak{S}_\nu$, and $a_x = 0$ for $x \in \mathfrak{S}_\mu d \mathfrak{S}_\nu$ by the argument as in the proof of [Ma1, Theorem 4.8]. Hence, by induction we can write φ as a linear combination of $\varphi_{\mu\nu}^d$ with $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha$ as required.

Finally, we have to show that $\{\varphi_{\mu\nu}^d \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$ is linearly independent. This follows from the fact that $\varphi_{\mu\nu}^d(m_\nu)$ is a linearly independent subset of M^μ , since the set $\{U_\mu g_w\}$ is linearly independent by the basis theorem for $Y_{r,n}$. \square

We write $M^{\nu*} = (M^\nu)^* = Y_{r,n} m_\nu$. As a corollary to Proposition 3.4, we have the next result.

Corollary 3.5. *Let $\mu, \nu \in \mathcal{C}_{r,n}$. Then $\text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$ and $M^{\nu*} \cap M^\mu$ are canonically isomorphic as \mathcal{R} -modules.*

Proof. Every homomorphism φ in $\text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$ is determined by $\varphi(m_\nu)$, and moreover, $\varphi(m_\nu) \in M^{\nu*} \cap M^\mu$ by Proposition 3.4. As a result, the map $\text{Hom}_{Y_{r,n}}(M^\nu, M^\mu) \rightarrow M^{\nu*} \cap M^\mu$ given by $\varphi \mapsto \varphi(m_\nu)$ is an isomorphism of \mathcal{R} -modules. \square

Remark 3.6. It is shown in [CR, 61.2] that whenever A is a quasi-hereditary algebra, $a \in A$ and J is an ideal of A then $\text{Hom}_A(aA, J) \cong Aa \cap J$. By [ChP1, Proposition 10] (see also [C1, Corollary 4.5]), $Y_{r,n}$ is quasi-Frobenius, so this gives another proof of Corollary 3.5.

Let $\mu, \nu \in \mathcal{C}_{r,n}$ and $\lambda \in \mathcal{P}_{r,n}$. We assume that $\alpha(\mu) = \alpha(\nu) = \alpha(\lambda)$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$, put

$$m_{ST} = \sum_{\mathfrak{s}, \mathfrak{t}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}},$$

where the sum is taken over all $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$ and $\nu(\mathfrak{t}) = T$.

Proposition 3.7. *Suppose that $\mu, \nu \in \mathcal{C}_{r,n}$ with $\alpha(\mu) = \alpha(\nu)$. Then the set*

$$\{m_{ST} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } T \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$$

is an \mathcal{R} -basis of $M^{\nu} \cap M^\mu$.*

Proof. Since

$$m_{ST} = \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} m_{T\mathfrak{s}}^* = \sum_{\substack{\mathfrak{t} \in \text{Std}(\lambda) \\ \nu(\mathfrak{t}) = T}} m_{S\mathfrak{t}},$$

we see that $m_{ST} \in M^{\nu*} \cap M^\mu$ by Lemma 3.2. Moreover, the elements m_{ST} are linearly independent since the basis elements $m_{\mathfrak{s}\mathfrak{t}}$ involved in the m_{ST} are distinct. Now suppose that $h \in M^{\nu*} \cap M^\mu$. Since $h \in Y_{r,n}$, we may express h with respect to the standard basis, that is, we may write $h = \sum r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ for some $r_{\mathfrak{s}\mathfrak{t}} \in \mathcal{R}$. Since $h \in M^\mu$, by Proposition 3.3 if $r_{\mathfrak{s}\mathfrak{t}} \neq 0$ then $\mu(\mathfrak{s}) \in \mathcal{T}_0^+(\lambda, \mu)$ for some $\lambda \in \mathcal{P}_{r,n}$ and $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}}$ whenever

$\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$. Similarly, since $h \in M^{\nu*}$, if $r_{\mathfrak{s}\mathfrak{t}} \neq 0$ then $\nu(\mathfrak{t}) \in \mathcal{T}_0^+(\lambda, \nu)$ for some $\lambda \in \mathcal{P}_{r,n}$ and $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}\mathfrak{t}'}$ whenever $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$. Consequently, if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}') \in \mathcal{T}_0^+(\lambda, \mu)$ and $\nu(\mathfrak{t}) = \nu(\mathfrak{t}') \in \mathcal{T}_0^+(\lambda, \nu)$, then $r_{\mathfrak{s}\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}'} = r_{\mathfrak{s}\mathfrak{t}'}$. This proves the proposition. \square

Definition 3.8. Suppose that $M_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} M^\mu$. We define the Yokonuma-Schur algebra $\text{YS}_n^r = \text{YS}_q(r, n)$ as the endomorphism algebra

$$\text{YS}_n^r = \text{End}_{Y_{r,n}}(M_n^r),$$

which is isomorphic to $\bigoplus_{\mu, \nu \in \mathcal{C}_{r,n}} \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$.

Let $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$. In view of Proposition 3.7, we can define $\varphi_{ST} \in \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$ by

$$\varphi_{ST}(m_\nu h) = m_{ST} h \quad (3.6)$$

for all $h \in Y_{r,n}$. We extend φ_{ST} to an element of YS_n^r by defining φ_{ST} to be zero on M^κ for $\nu \neq \kappa \in \mathcal{C}_{r,n}$. For any $\lambda \in \mathcal{P}_{r,n}$, let $\mathcal{T}_0^+(\lambda) = \bigcup_{\mu \in \mathcal{C}_{r,n}} \mathcal{T}_0^+(\lambda, \mu)$. We denote by $\text{YS}_{r,n}^{\triangleright \lambda}$ the \mathcal{R} -submodule of YS_n^r spanned by φ_{ST} such that $S, T \in \mathcal{T}_0^+(\alpha)$ with $\alpha \triangleright \lambda$. Then we have the next theorem.

Theorem 3.9. *The Yokonuma-Schur algebra YS_n^r is free as an \mathcal{R} -module with a basis*

$$\{\varphi_{ST} \mid S, T \in \mathcal{T}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$$

Moreover, this basis satisfies the following properties.

(i) *The \mathcal{R} -linear map $*$: $\text{YS}_n^r \rightarrow \text{YS}_n^r$ determined by $\varphi_{ST}^* = \varphi_{TS}$, for all $S, T \in \mathcal{T}_0^+(\lambda)$ and all $\lambda \in \mathcal{P}_{r,n}$, is an anti-automorphism of YS_n^r .*

(ii) *Let $T \in \mathcal{T}_0^+(\lambda)$ and $\varphi \in \text{YS}_n^r$. Then for each $V \in \mathcal{T}_0^+(\lambda)$, there exists $r_V = r_{V,T,\varphi} \in \mathcal{R}$ such that for all $S \in \mathcal{T}_0^+(\lambda)$, we have*

$$\varphi_{ST}\varphi \equiv \sum_{V \in \mathcal{T}_0^+(\lambda)} r_V \varphi_{SV} \pmod{\text{YS}_{r,n}^{\triangleright \lambda}}.$$

In particular, this basis $\{\varphi_{ST}\}$ is a cellular basis of YS_n^r .

Proof. The proof is similar to that of [Ma1, Theorem 4.14] and [DJM, Theorem 6.6]. By Corollary 3.5 and Proposition 3.7, the set $\{\varphi_{ST}\}$ is an \mathcal{R} -basis of YS_n^r . Next we need to verify (i) and (ii).

(i) Let $\lambda \in \mathcal{P}_{r,n}$ and $\mu, \nu \in \mathcal{C}_{r,n}$, and take $S \in \mathcal{T}_0^+(\lambda, \mu), T \in \mathcal{T}_0^+(\lambda, \nu)$. Then $\varphi_{ST}^*(m_\mu) = m_{TS} = (m_{ST})^* = (\varphi_{ST}(m_\nu))^*$. By the \mathcal{R} -linearity, we have $\varphi^*(m_\mu) = (\varphi(m_\nu))^*$ for any $\varphi \in \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$. Given $\varphi \in \text{Hom}_{Y_{r,n}}(M^\nu, M^\mu)$ and $\psi \in \text{Hom}_{Y_{r,n}}(M^\kappa, M^\lambda)$, we may assume that $\mu = \kappa$ since otherwise $\psi\varphi = 0$. Write $\varphi(m_\nu) = x_1 m_\nu$ and $\psi(m_\mu) = x_2 m_\mu$ for some $x_1, x_2 \in Y_{r,n}$. We have

$$\begin{aligned} (\psi\varphi)^*(m_\lambda) &= (\psi\varphi(m_\nu))^* = (x_2 x_1 m_\nu)^* = m_\nu x_1^* x_2^* \\ &= \varphi^*(m_\mu) x_2^* = \varphi^*(m_\mu x_2^*) = \varphi^* \psi^*(m_\lambda). \end{aligned}$$

Hence, $(\psi\varphi)^* = \varphi^*\psi^*$ and $*$ is an anti-automorphism.

(ii) Take $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$. We may assume that $\varphi \in \text{Hom}_{Y_{r,n}}(M^\kappa, M^\nu)$ for some $\kappa \in \mathcal{C}_{r,n}$ with $\alpha(\kappa) = \alpha(\nu)$. We have $\varphi(m_\kappa) = m_\nu h$ for some $h \in Y_{r,n}$. Then $\varphi_{ST}\varphi(m_\kappa) = m_{ST}h$. By Corollary 3.5, we see that $m_{ST}h \in M^{\kappa*} \cap M^\mu$. Hence by Proposition 3.7, we may write $m_{ST}h = \sum_{U,V} r_{UV}m_{UV}$, where $r_{UV} \in \mathcal{R}$, and the sum is over $U \in \mathcal{T}_0^+(\alpha, \mu)$ and $V \in \mathcal{T}_0^+(\alpha, \kappa)$ for some $\alpha \in \mathcal{P}_{r,n}$. By applying Theorem 2.5(ii), we can write $m_{ST}h$ as

$$m_{ST}h = \sum_{V \in \mathcal{T}_0^+(\lambda, \kappa)} r_V m_{SV} + \sum_{\substack{\alpha \in \mathcal{P}_{r,n} \\ \alpha \triangleright \lambda}} \sum_{\substack{U' \in \mathcal{T}_0^+(\alpha, \mu) \\ V' \in \mathcal{T}_0^+(\alpha, \kappa)}} r_{U'V'} m_{U'V'},$$

where $r_V, r_{U'V'} \in \mathcal{R}$. Therefore, we have

$$\varphi_{ST}\varphi \equiv \sum_{V \in \mathcal{T}_0^+(\lambda, \kappa)} r_V \varphi_{SV} \pmod{\text{YS}_{r,n}^{\triangleright \lambda}}.$$

We are done. \square

For each $\lambda \in \mathcal{P}_{r,n}$, let $T^\lambda = \lambda(\mathfrak{t}^\lambda)$. Then $T^\lambda \in \mathcal{T}_0^+(\lambda, \lambda)$ and T^λ is the unique semistandard λ -tableau of type λ . Moreover $\mathfrak{t} = \mathfrak{t}^\lambda$ is the unique element in $\text{Std}(\lambda)$ such that $\lambda(\mathfrak{t}) = T^\lambda$. Thus, $m_{T^\lambda T^\lambda} = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} = m_\lambda$, and $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$ is the identity map on M^λ .

The Weyl module W^λ is defined as the right YS_n^r -submodule of $\text{YS}_n^r/\text{YS}_{r,n}^{\triangleright \lambda}$ spanned by the image of φ_λ . For each $S \in \mathcal{T}_0^+(\lambda, \mu)$, we denote by φ_S the image of $\varphi_{T^\lambda S}$ in $\text{YS}_n^r/\text{YS}_{r,n}^{\triangleright \lambda}$. Then by Theorem 3.9, we see that W^λ , as an \mathcal{R} -module, is free with basis $\{\varphi_S \mid S \in \mathcal{T}_0^+(\lambda)\}$.

The Weyl module W^λ possesses an associative symmetric bilinear form, which is completely determined by the equation

$$\varphi_{T^\lambda S} \varphi_{T^\lambda T} \equiv \langle \varphi_S, \varphi_T \rangle \varphi_\lambda \pmod{\text{YS}_{r,n}^{\triangleright \lambda}}$$

for all $S, T \in \mathcal{T}_0^+(\lambda)$. Note that $\langle \varphi_S, \varphi_T \rangle = 0$ unless S and T are semistandard tableaux of the same type. Let $L^\lambda = W^\lambda / \text{rad} W^\lambda$, where $\text{rad} W^\lambda = \{x \in W^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in W^\lambda\}$.

Proposition 3.10. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field. Then for each $\lambda \in \mathcal{P}_{r,n}$, L^λ is an absolutely irreducible YS_n^r -module. Moreover, $\{L^\lambda \mid \lambda \in \mathcal{P}_{r,n}\}$ is a complete set of non-isomorphic irreducible YS_n^r -modules.*

Proof. For each $\lambda \in \mathcal{P}_{r,n}$, we have

$$\varphi_{T^\lambda T^\lambda} \varphi_{T^\lambda T^\lambda} \equiv \langle \varphi_{T^\lambda}, \varphi_{T^\lambda} \rangle \varphi_\lambda \pmod{\text{YS}_{r,n}^{\triangleright \lambda}}.$$

But since $\varphi_{T^\lambda T^\lambda} \varphi_{T^\lambda T^\lambda} = \varphi_\lambda$ is the identity map on M^λ , we see that $\langle \varphi_{T^\lambda}, \varphi_{T^\lambda} \rangle = 1$, and so L^λ is nonzero. Then the assertions follow from [GL, (3.4)]. \square

If $\lambda, \mu \in \mathcal{P}_{r,n}$, let $d_{\lambda\mu}$ denote the composition multiplicity of L^μ as a composition factor of W^λ . Then $(d_{\lambda\mu})_{\lambda, \mu \in \mathcal{P}_{r,n}}$ is the decomposition matrix of YS_n^r . The theory of cellular algebras [GL, (3.6)] yields the following result.

Corollary 3.11. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field. $(d_{\lambda\mu})_{\lambda, \mu \in \mathcal{P}_{r,n}}$ is unitriangular. That is, for $\lambda, \mu \in \mathcal{P}_{r,n}$, we have $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.*

Combining Proposition 3.10 with [GL, (3.10)], we have the next result.

Corollary 3.12. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field. The Yokonuma-Schur algebra YS_n^r is quasi-hereditary.*

Remark 3.13. For each $\lambda \in \mathcal{P}_{r,n}$ and for each $\mathfrak{t} \in \text{Std}(\lambda)$, let $m_{\mathfrak{t}} \in S^\lambda$ be the image of $m_{\mathfrak{t}\lambda\mathfrak{t}}$ under the map $Y_{r,n}/Y_{r,n}^{\triangleright\lambda}$. Then $\{m_{\mathfrak{t}}\} = \{\overline{m}_\lambda g_{d(\mathfrak{t})}\}$ gives an \mathcal{R} -basis of S^λ . For $T \in \mathcal{T}_0^+(\lambda, \mu)$, put $m_T = \sum_{\mathfrak{t}} q^{l(d(\mathfrak{t})) + l(d(\mathfrak{t}^\lambda))} m_{\mathfrak{t}} \in S^\lambda$, where the sum is taken over all \mathfrak{t} such that $\mu(\mathfrak{t}) = T$. Since m_T is the image of $m_{T^\lambda T}$, one obtains a well-defined map $\varphi_T \in \text{Hom}_{Y_{r,n}}(M^\mu, S^\lambda)$ by $\varphi_T(m_\mu) = m_T$, which is regarded as an element of $\text{Hom}_{Y_{r,n}}(M_n^r, S^\lambda)$ by extending by 0 outside. In a similar way as in [Ma1, Proposition 4.15], we see that W^λ is isomorphic to the YS_n^r -submodule of $\text{Hom}_{Y_{r,n}}(M_n^r, S^\lambda)$ with basis $\{\varphi_T \mid T \in \mathcal{T}_0^+(\lambda)\}$.

4 Schur functors

In this section, we will follow the approach in [HM2, §4.3] to define an exact functor from the category of YS_n^r -modules to the category of $Y_{r,n}$ -modules. For an algebra A , let $A\text{-mod}$ be the category of finite dimensional right A -modules.

Let $\dot{\mathcal{C}}_{r,n} = \mathcal{C}_{r,n} \cup \{\omega\}$, where ω is a dummy symbol. Set $M^\omega = Y_{r,n}$ and $\dot{M}_n^r = M_n^r \oplus M^\omega$. The extended Yokonuma-Schur algebra is the algebra

$$\dot{\text{YS}}_n^r = \text{End}_{Y_{r,n}}(\dot{M}_n^r).$$

Suppose that $\lambda \in \mathcal{P}_{r,n}$, and set $\mathcal{T}_0^+(\lambda, \omega) = \text{Std}(\lambda)$. Let $m_\omega = 1$ so that $M^\omega = m_\omega Y_{r,n}$. Let $\mathfrak{t}^\omega = 1$ and $m_{\mathfrak{t}^\omega} = 1$. We regard YS_n^r as a subalgebra of $\dot{\text{YS}}_n^r$ in the obvious way.

Extending (3.6), if $\lambda \in \mathcal{P}_{r,n}$, $\mu, \nu \in \dot{\mathcal{C}}_{r,n}$, and $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$, we define

$$\varphi_{ST}(m_\nu h) = m_{ST} h$$

for all $h \in Y_{r,n}$. Then $\varphi_{ST} \in \dot{\text{YS}}_n^r$. For each $\lambda \in \mathcal{P}_{r,n}$, set $\dot{\mathcal{T}}_0^+(\lambda) = \mathcal{T}_0^+(\lambda) \cup \mathcal{T}_0^+(\lambda, \omega) = \mathcal{T}_0^+(\lambda) \cup \text{Std}(\lambda)$.

Proposition 4.1. *The algebra $\dot{\text{YS}}_n^r$ is a cellular algebra with a cellular basis*

$$\{\varphi_{ST} \mid S, T \in \dot{\mathcal{T}}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$$

Moreover, if $\mathcal{R} = \mathbb{K}$ is a field, then $\dot{\text{YS}}_n^r$ is a quasi-hereditary algebra with Weyl modules $\{\dot{W}^\lambda \mid \lambda \in \mathcal{P}_{r,n}\}$ and simple modules $\{\dot{L}^\lambda \mid \lambda \in \mathcal{P}_{r,n}\}$.

Proof. By definition, YS_n^r is a subalgebra of $\dot{\text{YS}}_n^r$ and, as an \mathcal{R} -module,

$$\dot{\text{YS}}_n^r = \text{YS}_n^r \oplus \text{Hom}_{Y_{r,n}}(M^\omega, M_n^r) \oplus \text{Hom}_{Y_{r,n}}(M_n^r, M^\omega) \oplus \text{End}_{Y_{r,n}}(M^\omega, M^\omega).$$

For $\mu \in \dot{\mathcal{C}}_{r,n}$, there are isomorphisms of \mathcal{R} -modules $M^\mu \cong \text{Hom}_{Y_{r,n}}(M^\omega, M^\mu)$ given by $m_{S\mathfrak{t}} \mapsto \varphi_{S\mathfrak{t}}$, for $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$ with some $\lambda \in \mathcal{P}_{r,n}$. For $\nu \in \dot{\mathcal{C}}_{r,n}$, there are isomorphisms of \mathcal{R} -modules $M^{\nu*} \cong \text{Hom}_{Y_{r,n}}(M^\nu, M^\omega)$ given by $m_{\mathfrak{s}T} \mapsto \varphi_{\mathfrak{s}T}$, for $\mathfrak{s} \in \text{Std}(\lambda)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$ with some $\lambda \in \mathcal{P}_{r,n}$, where $m_{\mathfrak{s}T} = m_{T\mathfrak{s}}^*$. Therefore, the elements in the statement of this proposition give a basis of $\dot{\text{YS}}_n^r$ by Proposition 3.3 and Theorem 3.9.

Now suppose that $\mathcal{R} = \mathbb{K}$ is a field. Repeating the arguments from Theorem 3.9 and Proposition 3.10 shows that $\dot{\text{YS}}_n^r$ is a quasi-hereditary cellular algebra. \square

By Proposition 4.1, there exist Weyl modules \dot{W}^λ and simple modules $\dot{L}^\lambda = \dot{W}^\lambda / \text{rad} \dot{W}^\lambda$ for $\dot{\text{YS}}_n^r$, for each $\lambda \in \mathcal{P}_{r,n}$. Let $\{\varphi_S \mid S \in \dot{\mathcal{T}}_0^+(\lambda)\}$ be the basis of \dot{W}^λ . For each $\mu \in \mathcal{C}_{r,n}$, let φ_μ be the identity map on M^μ . We extend φ_μ to an element of YS_n^r by defining φ_μ to be zero on M^κ for $\mu \neq \kappa \in \mathcal{C}_{r,n}$. In particular, $\varphi_\mu = \varphi_{T^\mu T^\mu}$ if $\mu \in \mathcal{P}_{r,n}$. As an \mathcal{R} -module, every YS_n^r -module M has a weight space decomposition

$$M = \bigoplus_{\mu \in \mathcal{C}_{r,n}} M_\mu, \quad \text{where } M_\mu = M\varphi_\mu. \quad (4.1)$$

Set $\varphi_n^r = \sum_{\mu \in \mathcal{C}_{r,n}} \varphi_\mu$ and let φ_ω be the identity map on $M^\omega = Y_{r,n}$. Then φ_n^r is the identity element of YS_n^r and $\varphi_n^r + \varphi_\omega$ is the identity element of $\dot{\text{YS}}_n^r$. By definition, φ_n^r and φ_ω are both idempotents in $\dot{\text{YS}}_n^r$ and $\varphi_n^r \dot{\text{YS}}_n^r \varphi_n^r \cong \text{YS}_n^r$. Therefore, by [HM2, (2.10)], there are exact functors

$$\dot{F}_n^\omega : \dot{\text{YS}}_n^r\text{-mod} \rightarrow \text{YS}_n^r\text{-mod}, \quad \dot{G}_n^\omega : \text{YS}_n^r\text{-mod} \rightarrow \dot{\text{YS}}_n^r\text{-mod}$$

given by $\dot{F}_n^\omega(M) = M\varphi_n^r$ and $\dot{G}_n^\omega(N) = N \otimes_{\text{YS}_n^r} \varphi_n^r \dot{\text{YS}}_n^r$. By [HM, §2.4], there are functors $H_n^\omega := H_{\varphi_n^r}$, $O_n^\omega := O_{\varphi_n^r}$, $O_n^\omega := O^{\varphi_n^r}$ from $\dot{\text{YS}}_n^r\text{-mod}$ to $\text{YS}_n^r\text{-mod}$ such that $H_n^\omega(M) = M/O_n^\omega(M)$.

Lemma 4.2. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field. Then the functors \dot{F}_n^ω and \dot{G}_n^ω induce mutually inverse equivalences of categories between $\dot{\text{YS}}_n^r\text{-mod}$ and $\text{YS}_n^r\text{-mod}$. Moreover, $\dot{F}_n^\omega(\dot{W}^\lambda) \cong W^\lambda$ and $\dot{F}_n^\omega(\dot{L}^\lambda) \cong L^\lambda$ for all $\lambda \in \mathcal{P}_{r,n}$.*

Proof. Let M be a $\dot{\text{YS}}_n^r$ -module. Then, extending (4.1), M has a weight space decomposition

$$M = \bigoplus_{\mu \in \dot{\mathcal{C}}_{r,n}} M_\mu, \quad \text{where } M_\mu = M\varphi_\mu.$$

Then, essentially by definition, $\dot{F}_n^\omega(M) = \bigoplus_{\lambda \in \mathcal{P}_{r,n}} M_\lambda$. That is, \dot{F}_n^ω removes the ω -weight space of M . In particular, $\dot{F}_n^\omega(\dot{W}^\lambda) = W^\lambda$ and $\dot{F}_n^\omega(\dot{L}^\lambda) = L^\lambda$ for all $\lambda \in \mathcal{P}_{r,n}$. The fact that $\dot{F}_n^\omega(\dot{L}^\mu) = L^\mu$ for all $\mu \in \mathcal{P}_{r,n}$ implies that $O_n^\omega(M) = M$, $O_n^\omega(M) = 0$

for all $M \in \dot{\text{YS}}_n^r\text{-mod}$. Therefore, H_n^ω is the identity functor and $\dot{G}_n^\omega \cong H_n^\omega \circ \dot{G}_n^\omega$. Hence, this lemma is an application of the theory of quotient functors given in [HM2, Theorem 2.11]. \square

The identity map φ_ω on $Y_{r,n} = M^\omega$ is idempotent in $\dot{\text{YS}}_n^r$ and there is an isomorphism of \mathcal{R} -algebras $\varphi_\omega \dot{\text{YS}}_n^r \varphi_\omega \cong Y_{r,n}$. Therefore, by [HM2, (2.10)], there are functors

$$\dot{F}_n^r : \dot{\text{YS}}_n^r\text{-mod} \rightarrow Y_{r,n}\text{-mod}, \quad \dot{G}_n^r : Y_{r,n}\text{-mod} \rightarrow \dot{\text{YS}}_n^r\text{-mod}$$

given by $\dot{F}_n^r(M) = M\varphi_\omega = M_\omega$ and $\dot{G}_n^r(N) = N \otimes_{Y_{r,n}} \varphi_\omega \dot{\text{YS}}_n^r$.

Proposition 4.3. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field. Then there is an exact functor $F_n^r : \dot{\text{YS}}_n^r\text{-mod} \rightarrow Y_{r,n}\text{-mod}$ given by $F_n^r(M) = (M \otimes_{\dot{\text{YS}}_n^r} \varphi_n^r \dot{\text{YS}}_n^r) \varphi_\omega$, for $M \in \dot{\text{YS}}_n^r\text{-mod}$, such that if $\lambda, \mu \in \mathcal{P}_{r,n}$, then $F_n^r(W^\lambda) \cong S^\lambda$, and*

$$F_n^r(L^\mu) \cong \begin{cases} D^\mu & \text{if } \mu \in \mathcal{K}_{r,n}; \\ 0 & \text{if } \mu \notin \mathcal{K}_{r,n}. \end{cases}$$

Proof. By definition, $F_n^r = \dot{F}_n^r \circ \dot{G}_n^\omega$, so F_n^r is an exact functor from $\dot{\text{YS}}_n^r\text{-mod}$ to $Y_{r,n}\text{-mod}$. The functor \dot{F}_n^r is nothing more than projection onto the ω -weight space. Hence, if $\lambda \in \mathcal{P}_{r,n}$, then $\dot{F}_n^r(\dot{W}^\lambda)$ is spanned by the maps $\{\varphi_{\mathbf{t}} \mid \mathbf{t} \in \text{Std}(\lambda)\}$, since $\mathcal{T}_0^+(\lambda, \omega) = \text{Std}(\lambda)$. The map $\varphi_{\mathbf{t}} \mapsto m_{\mathbf{t}}$, for $\mathbf{t} \in \text{Std}(\lambda)$, defines an isomorphism $\dot{F}_n^r(\dot{W}^\lambda) \cong S^\lambda$ of $Y_{r,n}$ -modules. Therefore, $F_n^r(W^\lambda) \cong S^\lambda$ by Lemma 4.2.

By [HM2, Theorem 2.11], $F_n^r(L^\mu)$ is an irreducible $Y_{r,n}$ -module whenever it is nonzero. Using the fact that $F_n^r(W^\lambda) \cong S^\lambda$ and Corollary 3.11, a straightforward argument by induction on the dominance ordering shows that $F_n^r(L^\mu) \cong D^\mu$ if $\mu \in \mathcal{K}_{r,n}$ and that $F_n^r(L^\mu) = 0$ otherwise. \square

Since F_n^r is exact, we obtain the promised relationship between the decomposition numbers of $\dot{\text{YS}}_n^r$ and $Y_{r,n}$.

Corollary 4.4. *Suppose that $\mathcal{R} = \mathbb{K}$ is a field and that $\lambda \in \mathcal{P}_{r,n}$, $\mu \in \mathcal{K}_{r,n}$. Then $[S^\lambda : D^\mu] = [W^\lambda : L^\mu]$.*

Lemma 4.5. (A double centralizer property) *There are canonical isomorphisms of algebras such that $\dot{\text{YS}}_n^r = \text{End}_{Y_{r,n}}(\dot{M}_n^r)$ and $Y_{r,n} = \text{End}_{\dot{\text{YS}}_n^r}(\dot{M}_n^r)$. In particular, the functor \dot{F}_n^r is fully faithful on projectives.*

Proof. The first isomorphism is the definition of $\dot{\text{YS}}_n^r$, whereas the second follows directly from the definition of $\dot{\text{YS}}_n^r$ because

$$Y_{r,n} \cong \text{Hom}_{Y_{r,n}}(Y_{r,n}, Y_{r,n}) \cong \varphi_\omega \dot{\text{YS}}_n^r \varphi_\omega \cong \text{End}_{\dot{\text{YS}}_n^r}(\varphi_\omega \dot{\text{YS}}_n^r),$$

and $\varphi_\omega \dot{\text{YS}}_n^r \cong \dot{M}_n^r$ as a right $\dot{\text{YS}}_n^r$ -module. \square

Corollary 4.6. YS_n^r is a quasi-hereditary cover of $Y_{r,n}$ in the sense of Rouquier [R, Definition 4.34].

Proof. Recall that $\dot{M}_n^r \cong \varphi_\omega \dot{YS}_n^r$ is a projective \dot{YS}_n^r -module. Using the Morita equivalence between \dot{YS}_n^r and YS_n^r , we see that M_n^r is a projective YS_n^r -module. Because \dot{F}_n^r is fully faithful on projective modules by Lemma 4.5 and F_n^r is the composition of \dot{F}_n^r with an equivalence of categories, so is F_n^r . This implies that YS_n^r is a quasi-hereditary cover of $Y_{r,n}$ in the sense of Rouquier [R, Definition 4.34]. \square

5 Tilting modules

In this section, we introduce the tilting modules for YS_n^r and the closely related Young modules for $Y_{r,n}$ following [Ma2]. Throughout this section we assume that $\mathcal{R} = \mathbb{K}$ is a field.

5.1 Young modules and twisted Young modules

Recall from (4.1) that every YS_n^r -module has a weight space decomposition. Analogously, as a right YS_n^r -module, the regular representation of YS_n^r has a decomposition into a direct sum of left weight spaces

$$YS_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} Z^\mu, \quad \text{where } Z^\mu = \varphi_\mu YS_n^r \text{ for } \mu \in \mathcal{C}_{r,n}.$$

The next lemma gives some properties of the right YS_n^r -module Z^μ .

Lemma 5.1. *Assume that $\mu \in \mathcal{C}_{r,n}$. Then the following hold.*

(i) Z^μ is free as an \mathcal{R} -module with a basis

$$\{\varphi_{ST} \mid S \in \mathcal{T}_0^+(\lambda, \mu), T \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \nu \in \mathcal{C}_{r,n} \text{ and } \lambda \in \mathcal{P}_{r,n}\}.$$

(ii) Let $\mathcal{M}^\mu = \text{Hom}_{Y_{r,n}}(M_n^r, M^\mu)$. As right YS_n^r -modules, we have $Z^\mu \cong \mathcal{M}^\mu$.

(iii) As $Y_{r,n}$ -modules, we have $F_n^r(Z^\mu) \cong M^\mu$.

Proof. (i) It follows from Theorem 3.9.

(ii) It follows from (i).

(iii) We have

$$\begin{aligned} F_n^r(Z^\mu) &= \dot{F}_n^r \circ \dot{G}_n^\omega(\varphi_\mu YS_n^r) = \dot{F}_n^r(\varphi_\mu YS_n^r \otimes_{YS_n^r} \varphi_n^r \dot{YS}_n^r) \\ &\cong \dot{F}_n^r(\varphi_\mu \dot{YS}_n^r) \cong \text{Hom}_{Y_{r,n}}(Y_{r,n}, M^\mu) \cong M^\mu. \end{aligned}$$

We are done. \square

The next lemma claims that each Z^μ has a Weyl filtration.

Lemma 5.2. *Assume that $\mu \in \mathcal{C}_{r,n}$. Then Z^μ has a Weyl filtration*

$$Z^\mu = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

such that for each $1 \leq i \leq k$ there exists some $\lambda_i \in \mathcal{P}_{r,n}$ with $\alpha(\lambda_i) = \alpha(\mu)$ satisfying $M_i/M_{i+1} \cong W^{\lambda_i}$. Moreover, $\#\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \#\mathcal{T}_0^+(\lambda, \mu)$ for each $\lambda \in \mathcal{P}_{r,n}$ with $\alpha(\lambda) = \alpha(\mu)$.

Proof. Choose a total ordering $\{S_1, \dots, S_k\}$ on the set $\cup_{\lambda \in \mathcal{P}_{r,n}} \mathcal{T}_0^+(\lambda, \mu)$ such that $i > j$ whenever $\lambda_i \triangleright \lambda_j$, where $S_i \in \mathcal{T}_0^+(\lambda_i, \mu)$, $S_j \in \mathcal{T}_0^+(\lambda_j, \mu)$. For each $1 \leq i \leq k$, let M_i be the \mathcal{R} -submodule of Z^μ with basis $\{\varphi_{S_j T} \mid j \geq i \text{ and } T \in \mathcal{T}_0^+(\lambda_j)\}$. Then M_i is a YS_n^r -module by Theorem 3.9. Further, there is an isomorphism of YS_n^r -modules $W^{\lambda_i} \cong M_i/M_{i+1}$ given by $\varphi_T \mapsto \varphi_{S_i T} + M_{i+1}$ for $T \in \mathcal{T}_0^+(\lambda_i)$, because $M_i \cap \text{YS}_{r,n}^{\triangleright \lambda_i} \subseteq M_{i+1}$. Since YS_n^r is quasi-hereditary, $[Z^\mu : W^\lambda]$ is independent of the choice of Weyl filtration. \square

Applying the Schur functor F_n^r , by Proposition 4.3 and Lemma 5.1(iii), we also have the following result.

Corollary 5.3. *Assume that $\mu \in \mathcal{C}_{r,n}$. Then M^μ has a Specht filtration*

$$M^\mu = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

such that for each $1 \leq i \leq k$ there exists some $\lambda_i \in \mathcal{P}_{r,n}$ with $\alpha(\lambda_i) = \alpha(\mu)$ satisfying $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, $\#\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \#\mathcal{T}_0^+(\lambda, \mu)$ for each $\lambda \in \mathcal{P}_{r,n}$ with $\alpha(\lambda) = \alpha(\mu)$.

Since φ_μ is an idempotent in YS_n^r , Z^μ is a projective YS_n^r -module. Notice that if $\mathcal{T}_0(\lambda, \mu) \neq \emptyset$, then $\lambda \triangleright \mu$. Thus, W^λ appears in Z^μ only if $\lambda \triangleright \mu$. For each $\mu \in \mathcal{P}_{r,n}$, let P^μ be the projective cover of L^μ . Then by [Ma1, Lemma 2.16], P^μ has a filtration by Weyl modules in which W^λ appears with multiplicity $[P^\mu : W^\lambda] = [W^\lambda : L^\mu]$. From these, we can easily get the following lemma.

Lemma 5.4. *Assume that $\mu \in \mathcal{P}_{r,n}$. Then*

$$Z^\mu \cong P^\mu \oplus \bigoplus_{\lambda \triangleright \mu} c_{\lambda\mu} P^\lambda$$

for some non-negative integer $c_{\lambda\mu}$.

Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$. Then since $\mathcal{M}^\mu = \text{Hom}_{Y_{r,n}}(M_n^r, M^\mu)$, we can define a YS_n^r -module homomorphism $\Phi_{ST} : \mathcal{M}^\nu \rightarrow \mathcal{M}^\mu$ by $\Phi_{ST}(f) = \varphi_{ST} f$ for all $f \in \mathcal{M}^\nu$. In fact, these maps give a basis of $\text{Hom}_{\text{YS}_n^r}(\mathcal{M}^\nu, \mathcal{M}^\mu)$.

Lemma 5.5. *Suppose that $\mu, \nu \in \mathcal{C}_{r,n}$. Then $\text{Hom}_{\text{YS}_n^r}(\mathcal{M}^\nu, \mathcal{M}^\mu)$ is free as an \mathcal{R} -module with basis $\{\Phi_{ST} \mid S \in \mathcal{T}_0^+(\lambda, \mu), T \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$.*

For each $\lambda \in \mathcal{P}_{r,n}$, let $Y^\lambda = F_n^r(P^\lambda)$, which we call a Young module of $Y_{r,n}$.

Proposition 5.6. *Suppose that $\lambda \in \mathcal{P}_{r,n}$. Then the following hold.*

- (i) *Each Y^λ is an indecomposable $Y_{r,n}$ -module.*
- (ii) *If ν is another r -partition of n , then $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$.*
- (iii) *We have*

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\nu \triangleright \lambda} c_{\nu\lambda} Y^\nu.$$

- (iv) *The Young module Y^λ has a Specht filtration in which S^μ appears with multiplicity $[Y^\lambda : S^\mu] = [W^\mu : L^\lambda]$.*

Proof. (i) By Corollary 4.6, the functor F_n^r is fully faithful on projective modules, so $\text{End}_{Y_{r,n}}(Y^\lambda) \cong \text{End}_{Y_{r,n}}(P^\lambda)$ is a local ring since P^λ is indecomposable.

(ii) If $Y^\lambda \cong Y^\mu$, then $\text{Hom}_{Y_{r,n}}(Y^\lambda, Y^\mu)$ contains an isomorphism and this lifts to give an isomorphism $P^\mu \cong P^\lambda$, so $\lambda = \mu$.

(iii) Applying the Schur functor F_n^r , it follows from Lemma 5.1(iii) and Lemma 5.4.

(iv) Recall that P^λ has a Weyl filtration $P^\lambda = P_1 \supset P_2 \supset \cdots \supset P_k \supset P_{k+1} = 0$. Moreover, for each $\mu \in \mathcal{P}_{r,n}$, $\#\{1 \leq i \leq k \mid P_i/P_{i+1} \cong W^\mu\} = [P^\lambda : W^\mu] = [W^\mu : L^\lambda]$. Setting $Y_i = F_n^r(P_i)$, and using Proposition 4.3, gives a filtration of Y^λ with the required properties. \square

The next proposition identifies the projective Young modules, and its proof is similar to that of [HM2, Proposition 5.9].

Proposition 5.7. *Suppose that $\mu \in \mathcal{K}_{r,n}$. Then Y^μ is the projective cover of D^μ .*

Proof. Recall that P^μ is the projective cover of L^μ and $F_n^r(P^\mu) = Y^\mu$, $F_n^r(L^\mu) = D^\mu$ if $\mu \in \mathcal{K}_{r,n}$. Since F_n^r is exact, there is a surjective map $Y^\mu \twoheadrightarrow D^\mu$. Therefore, it suffices to show that Y^μ is projective since it is indecomposable by Proposition 5.6(i).

Recall that $\dot{Y}\dot{S}_n^r = \text{End}_{Y_{r,n}}(\dot{M}_n^r)$, where $\dot{M}_n^r = \dot{M}_n^r \oplus Y_{r,n}$. There is also a Schur functor \dot{F}_n^r from $\dot{Y}\dot{S}_n^r\text{-mod}$ to $Y_{r,n}\text{-mod}$ given by $\dot{F}_n^r(M) = M\varphi_\omega$. In particular, $\dot{F}_n^r(\dot{M}_n^r) \cong Y_{r,n}$ as right $Y_{r,n}$ -modules.

As a $\dot{Y}\dot{S}_n^r$ -module, $\dot{M}_n^r \cong \varphi_\omega \dot{Y}\dot{S}_n^r$. In particular, \dot{M}_n^r is a projective $\dot{Y}\dot{S}_n^r$ -module. If $\lambda \in \mathcal{P}_{r,n}$, let \dot{P}^λ be the projective cover of the irreducible $\dot{Y}\dot{S}_n^r$ -module \dot{L}^λ . The multiplicity of \dot{P}^λ as a summand of \dot{M}_n^r is equal to

$$\dim \text{Hom}_{\dot{Y}\dot{S}_n^r}(\dot{M}_n^r, \dot{L}^\lambda) = \dim \text{Hom}_{\dot{Y}\dot{S}_n^r}(\varphi_\omega \dot{Y}\dot{S}_n^r, \dot{L}^\lambda) = \dim \dot{L}^\lambda \varphi_\omega = \dim D^\lambda.$$

Consequently, $\dot{M}_n^r \cong \bigoplus_{\lambda \in \mathcal{K}_{r,n}} (\dim D^\lambda) \dot{P}^\lambda$ as a $\dot{Y}\dot{S}_n^r$ -module. By definition, $Y^\lambda = F_n^r(P^\lambda) = \dot{F}_n^r(\dot{P}^\lambda)$ for all $\lambda \in \mathcal{P}_{r,n}$. Therefore,

$$Y_{r,n} \cong \dot{F}_n^r(\dot{M}_n^r) \cong \bigoplus_{\lambda \in \mathcal{K}_{r,n}} (\dim D^\lambda) Y^\lambda$$

as a right $Y_{r,n}$ -module. The result follows. \square

Let $\mathcal{Z} = \mathbb{Z}[\frac{1}{r}][\dot{q}, \dot{q}^{-1}, \zeta]$, where \dot{q} is an indeterminate, and let $Y_{r,n}^{\mathcal{Z}}$ be the Yokonuma-Hecke algebra over \mathcal{Z} . It is easy to see that $Y_{r,n}^{\mathcal{Z}}$ has a \mathbb{Z} -algebra involution $'$ which is determined by

$$g'_i = g_i, \quad \dot{q}' = -\dot{q}^{-1}, \quad \text{and } t'_j = t_j$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

For each $\mu \in \mathcal{C}_{r,n}$, let $y_\mu = (x_\mu)' = \sum_{w \in \mathfrak{S}_\mu} (-\dot{q})^{-l(w)} g_w$, and let $n_\mu = U_\mu y_\mu$. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$. We define $n_{\mathfrak{st}} = g_{d(\mathfrak{s})}^* n_\lambda g_{d(\mathfrak{t})}$. Then $n_{\mathfrak{st}} = (m_{\mathfrak{st}})'$. Because $'$ is a \mathbb{Z} -algebra involution, $\{n_{\mathfrak{st}}\}$ is a cellular basis of $Y_{r,n}^{\mathcal{Z}}$ by Theorem 2.5. The ring \mathcal{R} is naturally a \mathbb{Z} -module under specialization; that is, \dot{q} acts on \mathcal{R} as multiplication by q . Because $Y_{r,n}$ is \mathcal{R} -free, this induces an isomorphism of \mathcal{R} -algebras $Y_{r,n} \cong Y_{r,n}^{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{R}$ via $g_i \mapsto g_i \otimes 1_{\mathcal{R}}$ ($1 \leq i \leq n-1$) and $t_j \mapsto t_j \otimes 1_{\mathcal{R}}$ ($1 \leq j \leq n$). Hereafter, we will identify the algebra $Y_{r,n}$ and $Y_{r,n}^{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{R}$ via the isomorphism above. Thus, we have the following result.

The Yokonuma-Hecke algebra $Y_{r,n}$ is free as an \mathcal{R} -module with a cellular basis $\{n_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$.

Now we can apply the general theory of cellular algebras. For each $\lambda \in \mathcal{P}_{r,n}$, we define the dual Specht module S_λ to be the right $Y_{r,n}$ -module $(n_\lambda + Y_{\triangleright \lambda}^{r,n})Y_{r,n}$, where $Y_{\triangleright \lambda}^{r,n} = (Y_{r,n}^{\triangleright \lambda})'$ is the two-sided ideal of $Y_{r,n}$ with basis $n_{\mathfrak{uv}}$ with $\mathfrak{u}, \mathfrak{v} \in \text{Std}(\nu)$ for various $\nu \in \mathcal{P}_{r,n}$ such that $\nu \triangleright \lambda$. Then S_λ is \mathcal{R} -free with basis $\{n_{\mathfrak{t}} \mid \mathfrak{t} \in \text{Std}(\lambda)\}$, where $n_{\mathfrak{t}} = n_{\mathfrak{t}\lambda} + Y_{\triangleright \lambda}^{r,n}$. Let $D_\lambda = S_\lambda / \text{rad} S_\lambda$, where $\text{rad} S_\lambda$ is the radical of the bilinear form on S_λ which is defined with respect to the cellular basis $\{n_{\mathfrak{st}}\}$.

For each $\mu \in \mathcal{C}_{r,n}$, let $N^\mu = n_\mu Y_{r,n}$. If $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$, we define

$$n_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} (-q)^{-l(d(\mathfrak{s})) - l(d(\mathfrak{t}))} n_{\mathfrak{st}}.$$

From the definition, we have $n_{S\mathfrak{t}} = (m_{S\mathfrak{t}})'$. Therefore, Proposition 3.3 and the usual specialization argument show that the following holds.

Corollary 5.8. *Suppose that $\mu \in \mathcal{C}_{r,n}$. Then N^μ is free as an \mathcal{R} -module with basis $\{n_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$.*

Let $\mu, \nu \in \mathcal{C}_{r,n}$ and $\lambda \in \mathcal{P}_{r,n}$. Suppose that $\alpha(\mu) = \alpha(\nu) = \alpha(\lambda)$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$, let

$$n_{ST} = \sum_{\substack{\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S, \nu(\mathfrak{t}) = T}} (-q)^{-l(d(\mathfrak{s})) - l(d(\mathfrak{t}))} n_{\mathfrak{st}}.$$

Now we can define the twisted Yokonuma-Schur algebra

$$\text{YS}_r^n = \text{End}_{Y_{r,n}}(N^n),$$

where $N^n = \bigoplus_{\mu \in \mathcal{C}_{r,n}} N^\mu$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$, we can also define the homomorphism φ'_{ST} by $\varphi'_{ST}(n_\alpha h) = \delta_{\alpha\nu} n_{ST} h$ for all $h \in Y_{r,n}$ and $\alpha \in \mathcal{C}_{r,n}$. Then

$\varphi'_{ST} \in \text{YS}_r^n$. The proof of the next proposition is in exactly the same way as that of [Ma2, Proposition 4.3], and we skip the details.

Proposition 5.9. (i) *The twisted Yokonuma-Schur algebra YS_r^n is free as an \mathcal{R} -module with a cellular basis*

$$\{\varphi'_{ST} \mid S, T \in \mathcal{T}_0^+(\lambda) \text{ for some } \mu, \nu \in \mathcal{C}_{r,n} \text{ and } \lambda \in \mathcal{P}_{r,n}\}.$$

(ii) *The twisted Yokonuma-Schur algebra YS_r^n is quasi-hereditary.*

(iii) *The \mathcal{R} -algebras YS_n^r and YS_r^n are canonically isomorphic.*

Let W_λ and L_λ ($\lambda \in \mathcal{P}_{r,n}$) be the Weyl modules and simple modules of YS_r^n , respectively; they are defined in exactly the same way as the corresponding modules for YS_n^r . As in Section 4, we can define an exact Schur functor F_r^n from $\text{YS}_r^n\text{-mod}$ to $Y_{r,n}\text{-mod}$. Moreover, we have $F_r^n(W_\lambda) \cong S_\lambda$, $F_r^n(L_\lambda) \cong D_\lambda$ and $[W_\lambda : L_\mu] = [S_\lambda : D_\mu]$ whenever $D_\mu \neq 0$.

For each $\lambda \in \mathcal{P}_{r,n}$, let P_λ be the projective cover of L_λ . Define $Y_\lambda = F_r^n(P_\lambda)$, which is called a twisted Young module. The next proposition can be proved in exactly the same way as in Proposition 5.6.

Proposition 5.10. *Let $\mu \in \mathcal{P}_{r,n}$. Then we have*

(i) *Each Y_μ is an indecomposable $Y_{r,n}$ -module.*

(ii) *If λ is another r -partition of n , then $Y_\lambda \cong Y_\mu$ if and only if $\lambda = \mu$.*

(iii)

$$N^\mu \cong Y_\mu \oplus \bigoplus_{\lambda \triangleright \mu} c_{\lambda\mu} Y_\lambda,$$

where the integers $c_{\lambda\mu}$ are the same as those appearing in Lemma 5.4.

(iv) *The twisted Young module Y_μ has a dual Specht filtration in which the number of subquotients equal to S_λ is $[W_\lambda : L_\mu]$.*

5.2 Non-degenerate bilinear forms

Suppose that $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ are two r -compositions. We say that λ dominates μ , and write $\lambda \supseteq \mu$, if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{i=1}^j \lambda_i^{(k)} \geq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{i=1}^j \mu_i^{(k)}$$

for all $1 \leq k \leq r$, and $j \geq 0$. If $\lambda \supseteq \mu$ and $\lambda \neq \mu$, then we write $\lambda \triangleright \mu$. If σ is a composition, its conjugate is the partition $\sigma' = (\sigma'_1, \sigma'_2, \dots)$, where σ'_i is the number of nodes in column i of the diagram of σ . If $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{C}_{r,n}$, its conjugate λ' is the r -partition $\lambda' = ((\lambda^{(r)})', \dots, (\lambda^{(1)})')$. Similarly, the conjugate of a standard λ -tableau $\mathbf{t} = (\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(r)})$ is the standard λ' -tableau $\mathbf{t}' = ((\mathbf{t}^{(r)})', \dots, (\mathbf{t}^{(1)})')$, where $(\mathbf{t}^{(k)})'$ is the tableau obtained by interchanging the rows and columns of $\mathbf{t}^{(k)}$.

If \mathbf{t} is a standard tableau, let $\mathbf{t}_{\downarrow k}$ be the subtableau of \mathbf{t} which contains $1, 2, \dots, k$, and let $\text{shape}(\mathbf{t}_{\downarrow k})$ be the associated multicomposition. Given two standard tableaux \mathbf{s} and \mathbf{t} , we say that \mathbf{s} dominates \mathbf{t} , and write $\mathbf{s} \supseteq \mathbf{t}$ if $\text{shape}(\mathbf{s}_{\downarrow k}) \supseteq \text{shape}(\mathbf{t}_{\downarrow k})$ for all $1 \leq k \leq n$. If $\mathbf{s} \supseteq \mathbf{t}$ and $\mathbf{s} \neq \mathbf{t}$, then we write $\mathbf{s} \triangleright \mathbf{t}$. Note that $\mathbf{s} \supseteq \mathbf{t}$ if and only if $\mathbf{t}' \supseteq \mathbf{s}'$. We extend the dominance order to pairs of standard tableaux by defining $(\mathbf{s}, \mathbf{t}) \supseteq (\mathbf{u}, \mathbf{v})$ if $\mathbf{s} \supseteq \mathbf{u}$ and $\mathbf{t} \supseteq \mathbf{v}$. We write $(\mathbf{s}, \mathbf{t}) \triangleright (\mathbf{u}, \mathbf{v})$ if $(\mathbf{s}, \mathbf{t}) \supseteq (\mathbf{u}, \mathbf{v})$ and $(\mathbf{s}, \mathbf{t}) \neq (\mathbf{u}, \mathbf{v})$.

For each $\lambda \in \mathcal{P}_{r,n}$, let $\mathbf{t}_\lambda = (\mathbf{t}^\lambda)'$. If $\mathbf{t} \in \text{Std}(\lambda)$, we define two elements $d(\mathbf{t})$ and $d'(\mathbf{t})$ in \mathfrak{S}_n by $\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t})$ and $\mathbf{t} = \mathbf{t}_\lambda d'(\mathbf{t})$. Conjugating either of the two equations shows that $d'(\mathbf{t}) = d(\mathbf{t}')$. Let $w_\lambda = d(\mathbf{t}_\lambda)$. In particular, we have $w_\lambda = w_\lambda'^{-1}$. Moreover, it is easy to see that $w_\lambda = d(\mathbf{t})d'(\mathbf{t})^{-1}$ and $l(w_\lambda) = l(d(\mathbf{t})) + l(d'(\mathbf{t}))$ for all $\mathbf{t} \in \text{Std}(\lambda)$.

Recall that there is a unique anti-automorphism $*$ on $Y_{r,n}$ such that $g_i^* = g_i$ for $1 \leq i \leq n-1$ and $t_j^* = t_j$ for $1 \leq j \leq n$. Given a right $Y_{r,n}$ -module M , we define its contragredient dual M^\circledast to be the dual module $\text{Hom}_{\mathcal{R}}(M, \mathcal{R})$ equipped with the right $Y_{r,n}$ -action $(\varphi h)(m) = \varphi(mh^*)$ for all $\varphi \in M^\circledast$, $h \in Y_{r,n}$ and $m \in M$. A module M is self-dual if $M \cong M^\circledast$. Equivalently, M is self-dual if and only if M possesses a non-degenerate associative bilinear form $\langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is associative if $\langle xh, y \rangle = \langle x, yh^* \rangle$ for all $x, y \in M$ and $h \in Y_{r,n}$.

Recall that the following Jucys-Murphy elements J_i ($1 \leq i \leq n$) in $Y_{r,n}$ have been introduced in [ChP] by induction

$$J_1 = 1 \quad \text{and} \quad J_i = g_{i-1} J_{i-1} g_{i-1}$$

for $i = 2, 3, \dots, n$. If $\lambda \in \mathcal{P}_{r,n}$ and $\mathbf{t} \in \text{Std}(\lambda)$, for $1 \leq k \leq n$, we define the residue of k in \mathbf{t} as the element $\text{res}_{\mathbf{t}}(k) = q^{2(j-i)}$ if k appears in row i and column j of some component $\mathbf{t}^{(s)}$ of \mathbf{t} . The following proposition is proved in [ER].

Proposition 5.11. (See [ER, Proposition 3].) *Suppose that $\lambda \in \mathcal{P}_{r,n}$ and \mathbf{s}, \mathbf{t} are two standard λ -tableaux. For each $1 \leq k \leq n$, there exist $r_{\mathbf{u}\mathbf{v}} \in \mathcal{R}$ such that*

$$m_{\mathbf{s}\mathbf{t}} J_k = \text{res}_{\mathbf{t}}(k) m_{\mathbf{s}\mathbf{t}} + \sum_{(\mathbf{u}, \mathbf{v})} r_{\mathbf{u}\mathbf{v}} m_{\mathbf{u}\mathbf{v}},$$

where the sum is over the pair $(\mathbf{u}, \mathbf{v}) \in \text{Std}^2(\mu) = \text{Std}(\mu) \times \text{Std}(\mu)$ such that $(\mathbf{u}, \mathbf{v}) \triangleright (\mathbf{s}, \mathbf{t})$ and $\alpha(\mu) = \alpha(\lambda)$.

Remark 5.12. There are two ways to define the dominance order on $\text{Std}^2(\mathcal{P}_{r,n}) = \{(\mathbf{s}, \mathbf{t}) \mid \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$. If $(\mathbf{s}, \mathbf{t}) \in \text{Std}^2(\lambda)$ and $(\mathbf{u}, \mathbf{v}) \in \text{Std}^2(\mu)$, then we define

$$(\mathbf{s}, \mathbf{t}) \blacktriangleright (\mathbf{u}, \mathbf{v}) \text{ if } \lambda \triangleright \mu, \text{ or } \lambda = \mu \text{ and } \mathbf{s} \supseteq \mathbf{u} \text{ and } \mathbf{t} \supseteq \mathbf{v}.$$

By definition, $(\mathbf{s}, \mathbf{t}) \supseteq (\mathbf{u}, \mathbf{v})$ implies that $(\mathbf{s}, \mathbf{t}) \blacktriangleright (\mathbf{u}, \mathbf{v})$, but the inverse is false in general. In fact, it is proved in [ER] that the above equality holds under the dominance order \blacktriangleright . But it is easy to see that the above equality still holds under the stronger dominance order \supseteq . Besides, the proof of Proposition 5.11 essentially reduces to the case of $r = 1$,

from which we can easily get the second condition $\alpha(\mu) = \alpha(\lambda)$ in the summation. These facts are crucial to the following arguments.

Let $\mathcal{K} = \mathbb{Q}(q, \zeta)$. Set $\text{Std}(\mathcal{P}_{r,n}) = \bigcup_{\lambda \in \mathcal{P}_{r,n}} \text{Std}(\lambda)$. For each $\lambda \in \mathcal{P}_{r,n}$ and $\mathfrak{t} \in \text{Std}(\lambda)$, following [Ma1, 3.31] we define $F_{\mathfrak{t}} \in Y_{r,n}^{\mathcal{K}}$ by

$$F_{\mathfrak{t}} = \prod_{k=1}^n \prod_{\substack{\mathfrak{s} \in \text{Std}(\mathcal{P}_{r,n}) \\ \text{res}_{\mathfrak{s}}(k) \neq \text{res}_{\mathfrak{t}}(k)}} \frac{J_k - \text{res}_{\mathfrak{s}}(k)}{\text{res}_{\mathfrak{t}}(k) - \text{res}_{\mathfrak{s}}(k)}.$$

Given two standard λ -tableaux \mathfrak{s} and \mathfrak{t} , set $f_{\mathfrak{st}} = F_{\mathfrak{s}} m_{\mathfrak{st}} F_{\mathfrak{t}}$ and $g_{\mathfrak{st}} = F_{\mathfrak{s}'} n_{\mathfrak{st}} F_{\mathfrak{t}'}$. From the definition, we have $(\text{res}_{\mathfrak{t}}(k))' = \text{res}_{\mathfrak{t}'}(k)$ for all tableaux \mathfrak{t} and k . This implies that $F_{\mathfrak{t}'} = F_{\mathfrak{t}}$ and hence that $g_{\mathfrak{st}} = f_{\mathfrak{st}'}^{\mathcal{K}}$ in $Y_{r,n}^{\mathcal{K}}$.

In view of Proposition 5.11, we can apply the results of [Ma3, Section 3] and get the following results. We leave the details to the reader.

Proposition 5.13. *In $Y_{r,n}^{\mathcal{K}}$, we have*

- (i) $m_{\mathfrak{st}} = f_{\mathfrak{st}} + \sum_{(\mathfrak{u}, \mathfrak{v})} r_{\mathfrak{uv}} f_{\mathfrak{uv}}$, where $r_{\mathfrak{uv}} \in \mathcal{K}$ and the sum is over the pair $(\mathfrak{u}, \mathfrak{v}) \in \text{Std}^2(\mu)$ such that $r_{\mathfrak{uv}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \triangleright (\mathfrak{s}, \mathfrak{t})$ and $\alpha(\mu) = \alpha(\lambda)$.
- (ii) $n_{\mathfrak{st}} = g_{\mathfrak{st}} + \sum_{(\mathfrak{a}, \mathfrak{b})} r_{\mathfrak{ab}} g_{\mathfrak{ab}}$, where $r_{\mathfrak{ab}} \in \mathcal{K}$ and the sum is over the pair $(\mathfrak{a}, \mathfrak{b}) \in \text{Std}^2(\nu)$ such that $r_{\mathfrak{ab}} \neq 0$ only if $(\mathfrak{a}, \mathfrak{b}) \triangleright (\mathfrak{s}, \mathfrak{t})$ and $\alpha(\nu) = \alpha(\lambda)$.
- (iii) Suppose that $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \mathfrak{v}$ are standard tableaux. Then $f_{\mathfrak{st}} g_{\mathfrak{uv}} = 0$ unless $\mathfrak{t} = \mathfrak{u}'$.

Using Proposition 5.13 we can easily get the next lemma.

Lemma 5.14. *Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Suppose that \mathfrak{s} and \mathfrak{t} are standard λ -tableaux and that \mathfrak{u} and \mathfrak{v} are standard μ -tableaux such that $m_{\mathfrak{st}} n_{\mathfrak{uv}} \neq 0$. Then $\mathfrak{u}' \triangleright \mathfrak{t}$ and $\alpha(\mu') = \alpha(\lambda)$.*

Recall that $\{t_1^{k_1} \cdots t_n^{k_n} g_w \mid k_1, \dots, k_n \in \mathbb{Z}/r\mathbb{Z} \text{ and } w \in \mathfrak{S}_n\}$ is an \mathcal{R} -basis of $Y_{r,n}$. We can define an \mathcal{R} -linear map $\tau : Y_{r,n} \rightarrow \mathcal{R}$ by

$$\tau(t_1^{k_1} \cdots t_n^{k_n} g_w) = \begin{cases} 1 & \text{if } k_1 \equiv k_2 \equiv \cdots \equiv 0 \pmod{r} \text{ and } w = 1; \\ 0 & \text{otherwise.} \end{cases}$$

This map was introduced in [ChP, Proposition 10] and was shown to be a trace form; that is, $\tau(ab) = \tau(ba)$ for all $a, b \in Y_{r,n}$. Moreover, we have

$$\tau(t_1^{k_1} \cdots t_n^{k_n} g_w g_{w'} t_1^{l_1} \cdots t_n^{l_n}) = \begin{cases} 1 & \text{if } w^{-1} = w' \text{ and } k_i + l_i \equiv 0 \pmod{r} \text{ for } 1 \leq i \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we get that $\tau(h^*) = \tau(h)$ for all $h \in Y_{r,n}$.

Now define a symmetric associative bilinear form $\langle \cdot, \cdot \rangle$ on $Y_{r,n}$ by $\langle h_1, h_2 \rangle = \tau(h_1 h_2^*)$. We have the following crucial result.

Theorem 5.15. *Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Suppose that $(\mathfrak{s}, \mathfrak{t})$ is a pair of standard λ -tableaux and that $(\mathfrak{u}, \mathfrak{v})$ is a pair of standard μ -tableaux. Then we have*

$$\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{u}\mathfrak{v}} \rangle = \begin{cases} 1 & \text{if } (\mathfrak{u}', \mathfrak{v}') = (\mathfrak{s}, \mathfrak{t}); \\ 0 & \text{if } (\mathfrak{u}', \mathfrak{v}') \not\supseteq (\mathfrak{s}, \mathfrak{t}). \end{cases}$$

Proof. Suppose first that $\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{u}\mathfrak{v}} \rangle \neq 0$. Now $\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{u}\mathfrak{v}} \rangle = \tau(m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{v}\mathfrak{u}})$, so $m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{v}\mathfrak{u}} \neq 0$; hence $\mathfrak{v}' \supseteq \mathfrak{t}$ by Lemma 5.14. Since τ is a trace form and $\tau(h^*) = \tau(h)$ for all $h \in Y_{r,n}$, we also have $\tau(m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{v}\mathfrak{u}}) = \tau(n_{\mathfrak{v}\mathfrak{u}}m_{\mathfrak{s}\mathfrak{t}}) = \tau(m_{\mathfrak{t}\mathfrak{s}}n_{\mathfrak{u}\mathfrak{v}})$; hence $m_{\mathfrak{t}\mathfrak{s}}n_{\mathfrak{u}\mathfrak{v}} \neq 0$ and $\mathfrak{u}' \supseteq \mathfrak{s}$ by Lemma 5.14. Therefore, if $\langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{u}\mathfrak{v}} \rangle \neq 0$, then $(\mathfrak{u}', \mathfrak{v}') \supseteq (\mathfrak{s}, \mathfrak{t})$.

Now assume that $(\mathfrak{u}', \mathfrak{v}') = (\mathfrak{s}, \mathfrak{t})$. Then $g_{w_\lambda} = g_{d(\mathfrak{t})}g_{d(\mathfrak{v})}^* = g_{d(\mathfrak{s})}g_{d(\mathfrak{s}')}^*$. Therefore, we have

$$\begin{aligned} \langle m_{\mathfrak{s}\mathfrak{t}}, n_{\mathfrak{s}'\mathfrak{t}'} \rangle &= \tau(m_{\mathfrak{s}\mathfrak{t}}n_{\mathfrak{t}'\mathfrak{s}'}) = \tau(g_{d(\mathfrak{s})}^*m_\lambda g_{d(\mathfrak{t})}g_{d(\mathfrak{t}')}^*n_{\lambda'}g_{d(\mathfrak{s}')}) \\ &= \tau(g_{d(\mathfrak{s}')}g_{d(\mathfrak{s})}^*m_\lambda g_{w_\lambda}n_{\lambda'}) = \tau(g_{w_\lambda}^*m_\lambda g_{w_\lambda}n_{\lambda'}). \end{aligned}$$

Since $w_\lambda^{-1}\mathfrak{S}_\lambda \cap \mathfrak{S}_{\lambda'} = \{1\}$ and $w_\lambda = w_{\lambda'}^{-1}$ is a distinguished $(\mathfrak{S}_\lambda, \mathfrak{S}_{\lambda'})$ -double coset representative, we have

$$\begin{aligned} g_{w_\lambda}^*m_\lambda g_{w_\lambda}n_{\lambda'} &= \sum_{\substack{u \in \mathfrak{S}_\lambda \\ v \in \mathfrak{S}_{\lambda'}}} q^{l(u)} g_{w_\lambda^{-1}} g_u g_{w_\lambda} (-q)^{-l(v)} g_v \\ &= \sum_{\substack{u \in \mathfrak{S}_\lambda \\ v \in \mathfrak{S}_{\lambda'}}} (-1)^{l(v)} q^{l(u)-l(v)} g_{w_\lambda^{-1}} g_{uw_\lambda v}. \end{aligned}$$

Thus, we get $\tau(g_{w_\lambda}^*m_\lambda g_{w_\lambda}n_{\lambda'}) = 1$. □

The next corollary can be proved in exactly the same way as in [Ma2, Corollary 5.7] using Theorem 5.15, which justify the term dual Specht module.

Corollary 5.16. *Suppose that $\lambda \in \mathcal{P}_{r,n}$. Then $S^{\lambda'} \cong S_\lambda^\circ$.*

If $S = (S^{(1)}, \dots, S^{(r)})$ is a λ -tableau of type μ , we define the conjugate of S by $S = ((S^{(r)})', \dots, (S^{(1)})')$ which is a λ' -tableau of type μ , where $(S^{(j)})'$ is the tableau obtained by interchanging the rows and columns of $S^{(j)}$ for each j . A λ -tableau S is called column semistandard if S' is semistandard. For $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \mathcal{C}_{r,n}$, let $\mathcal{T}^{cs}(\lambda, \mu) = \{S \mid S' \in \mathcal{T}_0^+(\lambda', \mu)\}$.

The proof of the next lemma is in exactly the same way as that of [Ma2, Lemma 5.8] by making use of Lemma 5.14. We skip the details.

Lemma 5.17. *Suppose that $\mu \in \mathcal{C}_{r,n}, \lambda \in \mathcal{P}_{r,n}$ and that $m_\mu n_{\mathfrak{u}\mathfrak{v}} \neq 0$ or $n_\mu m_{\mathfrak{u}\mathfrak{v}} \neq 0$ for some standard λ -tableaux \mathfrak{u} and \mathfrak{v} . Then $\mu(\mathfrak{u})$ is column semistandard and $\alpha(\lambda') = \alpha(\mu)$; that is, $\mu(\mathfrak{u}) \in \mathcal{T}^{cs}(\lambda, \mu)$.*

Remark 5.18. As mentioned in [HM1, p. 15], it is unfortunate that Mathas confused the two partial orders \supseteq and \blacktriangleright on $\text{Std}^2(\mathcal{P}_{r,n})$ in [Ma2] and [Ma3]. Anyhow, we can adapt the approach in [Ma3, Section 3] to get Proposition 5.13 and then Lemma 5.14 and 5.17. We leave the details to the reader; see also [HM1, Section 2] for details.

If $S \in \mathcal{T}^{cs}(\lambda, \mu)$, let \dot{S} be the unique standard λ -tableau such that $\mu(\dot{S}) = S$ and $d(\dot{S})$ is a distinguished $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$ -double coset representative; that is, $d(\dot{S})$ is the unique element of minimal length in its double coset.

Proposition 5.19. *Suppose that $\mu \in \mathcal{C}_{r,n}$. Then M^μ is free as an \mathcal{R} -module with basis $\{m_\mu n_{\dot{S}\mathfrak{t}} \mid S \in \mathcal{T}^{cs}(\lambda, \mu) \text{ and } \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$ and N^μ is free as an \mathcal{R} -module with basis $\{n_\mu m_{\dot{S}\mathfrak{t}} \mid S \in \mathcal{T}^{cs}(\lambda, \mu) \text{ and } \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$.*

Proof. We only prove the claim for M^μ . Recall that $\{n_{\mathfrak{st}}\}$ is an \mathcal{R} -basis of $Y_{r,n}$, so M^μ is spanned by the elements $m_\mu n_{\mathfrak{st}}$, where $(\mathfrak{s}, \mathfrak{t}) \in \text{Std}^2(\mathcal{P}_{r,n})$. Furthermore, if $m_\mu n_{\mathfrak{st}} \neq 0$ then $\mu(\mathfrak{s})$ is column semistandard and $\alpha(\lambda') = \alpha(\mu)$ if $(\mathfrak{s}, \mathfrak{t}) \in \text{Std}^2(\lambda)$ by Lemma 5.17. Hence, M^μ is spanned by the elements $m_\mu n_{\mathfrak{st}}$, where $\mu(\mathfrak{s})$ is column semistandard and $(\mathfrak{s}, \mathfrak{t}) \in \text{Std}^2(\lambda)$ for various $\lambda \in \mathcal{P}_{r,n}$ with $\alpha(\lambda') = \alpha(\mu)$.

For each such element $m_\mu n_{\mathfrak{st}}$, where $(\mathfrak{s}, \mathfrak{t}) \in \text{Std}^2(\nu)$ with $\alpha(\nu') = \alpha(\mu)$. Since $\mu(\mathfrak{s}) = S$ is column semistandard, we choose $\dot{S} \in \text{Std}(\nu)$ as above and get that $\mu(\mathfrak{t}^\nu d(\mathfrak{s})) = \mu(\mathfrak{t}^\nu d(\dot{S}))$. Thus, $d(\mathfrak{s})$ and $d(\dot{S})$ lie in the same $(\mathfrak{S}_\nu, \mathfrak{S}_\mu)$ -double coset. By definition $d(\dot{S})$ is the unique element of minimal length in its double coset, therefore we get $m_\mu g_{d(\mathfrak{s})}^* n_\nu = \pm q^a m_\mu g_{d(\dot{S})}^* n_\nu$ for some integer a . Because M^μ is \mathcal{R} -free and the number of elements in our spanning set is exactly the rank of M^μ , thus we have proved the first claim. The second statement can be proved in a similar way. \square

Combining Lemma 5.17 and Proposition 5.19 we have the next result.

Corollary 5.20. *Suppose that $\mu \in \mathcal{C}_{r,n}$, $\lambda \in \mathcal{P}_{r,n}$ and that \mathfrak{u} and \mathfrak{v} are two standard λ -tableaux. Then $m_\mu n_{\mathfrak{uv}} \neq 0$ if and only if $\mu(\mathfrak{u})$ is column semistandard and $\alpha(\lambda') = \alpha(\mu)$. Similarly, $n_\mu m_{\mathfrak{uv}} \neq 0$ if and only if $\mu(\mathfrak{u})$ is column semistandard and $\alpha(\lambda') = \alpha(\mu)$.*

Using Proposition 5.19 we can get the following result by repeating the argument of Lemma 5.2.

Corollary 5.21. *Suppose that $\mu \in \mathcal{C}_{r,n}$. Then there exist filtrations*

$$M^\mu = H^1 \supset H^2 \supset \cdots \supset H^k \supset H^{k+1} = 0 \text{ and } N^\mu = H_1 \supset H_2 \supset \cdots \supset H_k \supset H_{k+1} = 0$$

of M^μ and N^μ , respectively, and multipartitions $\lambda_1, \dots, \lambda_k$ such that $\mu' \supseteq \lambda_i$, $H^i/H^{i+1} \cong S_{\lambda_i}$ and $H_i/H_{i+1} \cong S^{\lambda_i}$ for $1 \leq i \leq k$. Moreover, for any $\lambda \in \mathcal{P}_{r,n}$ we have $\#\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \#\mathcal{T}^{cs}(\lambda, \mu)$.

Now we can define a bilinear form $\langle \cdot, \cdot \rangle_\mu$ on M^μ by $\langle m_{S\mathfrak{t}}, m_\mu n_{\dot{U}\mathfrak{v}} \rangle_\mu = \langle m_{S\mathfrak{t}}, n_{\dot{U}\mathfrak{v}} \rangle$, where $m_{S\mathfrak{t}}$ and $m_\mu n_{\dot{U}\mathfrak{v}}$ run over the bases of Proposition 3.3 and 5.19, respectively.

The next proposition can be proved in exactly the same way as in [Ma2, Proposition 5.13]. We omit the details and leave them to the reader.

Proposition 5.22. *Suppose that $\mu \in \mathcal{C}_{r,n}$. Then $\langle \cdot, \cdot \rangle_\mu$ is a non-degenerate associative bilinear form on M^μ . In particular, M^μ is self-dual. Similarly, N^μ is self-dual.*

By induction and using Proposition 5.6 and 5.10 we can get the next result.

Corollary 5.23. *Let $\lambda \in \mathcal{P}_{r,n}$. Then the Young module Y^λ and twisted Young module Y_λ are both self-dual.*

5.3 Tilting modules

Recall that a YS_n^r -module T is a tilting module if it has both a filtration by Weyl modules W^λ ($\lambda \in \mathcal{P}_{r,n}$) and a filtration by dual Weyl modules. Since YS_n^r is quasi-hereditary, by [Ri], for each $\lambda \in \mathcal{P}_{r,n}$ there exists a unique indecomposable tilting module T^λ such that $[T^\lambda : W^\lambda] = 1$ and $[T^\lambda : W^\mu] \neq 0$ only if $\lambda \geq \mu$. Moreover, any tilting module T can be written as a direct sum of these T^λ 's. The T^λ are the partial tilting modules of YS_n^r . A full tilting module for YS_n^r is any tilting module which contain every T^λ ($\lambda \in \mathcal{P}_{r,n}$) as a direct summand.

For each $\nu \in \mathcal{C}_{r,n}$, let $\theta_\nu \in \text{Hom}_{Y_{r,n}}(Y_{r,n}, N^\nu)$ be the map given by $\theta_\nu(h) = n_\nu h$ for all $h \in Y_{r,n}$ and define $E^\nu = \dot{F}_n^\omega(\theta_\nu \text{YS}_n^r)$. Since E^ν , by definition, is the set of maps from M_n^r to N^ν which factor through θ_ν , we get that E^ν is a right YS_n^r -module.

Definition 5.24. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mu, \nu \in \mathcal{C}_{r,n}$. For $S \in \mathcal{T}^{cs}(\lambda, \nu)$ and $T \in \mathcal{T}_0^+(\lambda, \mu)$ let θ_{ST} be the homomorphism in E^ν determined by $\theta_{ST}(m_\alpha h) = \delta_{\alpha\mu} n_\nu m_{\dot{S}T} h$ for all $h \in Y_{r,n}$ and all $\alpha \in \mathcal{C}_{r,n}$.

Proposition 5.25. *Let $\nu \in \mathcal{C}_{r,n}$. Then E^ν is free as an \mathcal{R} -module with basis*

$$\{\theta_{ST} \mid S \in \mathcal{T}^{cs}(\lambda, \nu) \text{ and } T \in \mathcal{T}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$$

Proof. Let $\dot{E}^\nu = \theta_\nu \text{YS}_n^r$. Then \dot{E}^ν is a right YS_n^r -module and $E^\nu = \dot{F}_n^\omega(\dot{E}^\nu)$. By Proposition 4.1, \dot{E}^ν is spanned by the maps $\theta_\nu \varphi_{ST}$, where $S \in \mathcal{T}_0^+(\lambda, \sigma)$ and $T \in \mathcal{T}_0^+(\lambda, \mu)$ for various $\lambda \in \mathcal{P}_{r,n}$ and $\sigma, \mu \in \dot{\mathcal{C}}_{r,n}$. By definition, $\theta_\nu \varphi_{ST} = 0$ unless $\sigma = \omega$, that is, S is a standard λ -tableau; so \dot{E}^ν is spanned by the elements $\theta_\nu \varphi_{\mathfrak{s}T}$ with $\mathfrak{s} \in \text{Std}(\lambda)$ and $T \in \mathcal{T}_0^+(\lambda, \mu)$ for $\lambda \in \mathcal{P}_{r,n}$ and $\mu \in \dot{\mathcal{C}}_{r,n}$. Furthermore, $\theta_\nu \varphi_{\mathfrak{s}T}(m_\mu h) = n_\nu m_{\mathfrak{s}T} h$ for all $h \in Y_{r,n}$. Thus, $\theta_\nu \varphi_{\mathfrak{s}T} \neq 0$ if and only if $\nu(\mathfrak{s})$ is column semistandard and $\alpha(\lambda') = \alpha(\nu)$ by Corollary 5.20, and in this case $\theta_\nu \varphi_{\mathfrak{s}T} = \pm q^a \theta_{ST}$ for some $a \in \mathbb{Z}$ and $S = \nu(\mathfrak{s})$. Hence these elements $\{\theta_{ST} \mid S \in \mathcal{T}^{cs}(\lambda, \nu) \text{ and } T \in \dot{\mathcal{T}}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$ span \dot{E}^ν .

On the other hand, the elements $\{\theta_{ST}\}$ are linearly independent by Proposition 5.19, so they are a basis of \dot{E}^ν . Since the functor \dot{F}_n^ω removes the ω -weight space of

\dot{E}^ν , therefore \dot{F}_n^ω maps the basis $\{\theta_{ST}\}$ of \dot{E}^ν to the elements stated in the proposition, or to zero if $\mu = \omega$. Hence, $\{\theta_{ST} \mid S \in \mathcal{T}^{cs}(\lambda, \nu) \text{ and } T \in \mathcal{T}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}$ is an \mathcal{R} -basis of E^ν . \square

The next proposition can be proved in exactly the same way as in [Ma2, Theorem 2.5] by using Proposition 5.25. We skip the details.

Proposition 5.26. *Let $\nu \in \mathcal{C}_{r,n}$. Then E^ν admits a YS_n^r -module filtration $E^\nu = E_1 \supset E_2 \supset \cdots \supset E_k \supset E_{k+1} = 0$ such that $E_i/E_{i+1} \cong W^{\lambda_i}$ for some $\lambda_1, \dots, \lambda_k \in \mathcal{P}_{r,n}$ and $\nu' \triangleright \lambda_i$ for all $1 \leq i \leq k$. Moreover, if $\lambda \in \mathcal{P}_{r,n}$, then $\#\{1 \leq i \leq k \mid \lambda_i = \lambda\} = \#\mathcal{T}^{cs}(\lambda, \nu)$.*

From this proposition we can easily get the next corollary.

Corollary 5.27. *Suppose that $\lambda, \mu \in \mathcal{P}_{r,n}$. Then $[E^\lambda : W^{\lambda'}] = 1$ and $[E^\lambda : W^\mu] \neq 0$ only if $\lambda' \triangleright \mu$.*

Definition 5.28. Suppose that $\lambda \in \mathcal{P}_{r,n}$ and $\mu, \nu \in \mathcal{C}_{r,n}$. For $A \in \mathcal{T}_0^+(\lambda, \nu)$ and $B \in \mathcal{T}^{cs}(\lambda, \mu)$ let θ'_{AB} be the homomorphism determined by $\theta'_{AB}(m_\alpha h) = \delta_{\alpha\mu} n_{A\dot{B}} m_\mu h$ for all $h \in Y_{r,n}$ and all $\alpha \in \mathcal{C}_{r,n}$.

Proposition 5.29. *Let $\nu \in \mathcal{C}_{r,n}$. Then E^ν is free as an \mathcal{R} -module with basis*

$$\{\theta'_{AB} \mid A \in \mathcal{T}_0^+(\lambda, \nu) \text{ and } B \in \mathcal{T}^{cs}(\lambda, \mu) \text{ for some } \lambda \in \mathcal{P}_{r,n} \text{ and } \mu \in \mathcal{C}_{r,n}\}.$$

Proof. We first show that $\theta'_{AB} \in E^\nu$. By Corollary 5.8, $n_{A\dot{B}} = n_\nu x$ for some $x \in Y_{r,n}$. Therefore, $\theta'_{AB}(m_\mu h) = n_{A\dot{B}} m_\mu h = n_\nu x m_\mu h = \theta_\nu(x m_\mu h)$. That is, θ'_{AB} factors through θ_ν so that $\theta'_{AB} \in E^\nu$ as claimed. Moreover, these elements stated in the proposition are linearly independent by applying $*$ to Proposition 5.19. Therefore, these elements $\{\theta'_{AB}\}$ give a basis of E^ν by counting dimensions using Proposition 5.25. \square

The contragredient dual E^\circledast of a YS_n^r -module E can be defined in exactly the same way as that of $Y_{r,n}$ -modules. Again, we say that E is self-dual if $E \cong E^\circledast$.

Using the two bases $\{\theta_{ST}\}$ and $\{\theta'_{AB}\}$ in Proposition 5.25 and 5.29, we now define a bilinear form $\{\cdot, \cdot\}_\nu$ on E^ν by

$$\langle \theta_{ST}, \theta'_{AB} \rangle_\nu = \begin{cases} \langle m_{\dot{S}T}, n_{A\dot{B}} \rangle & \text{if Type}(T) = \text{Type}(B); \\ 0 & \text{otherwise.} \end{cases}$$

The next theorem can be proved in exactly the same way as in [Ma2, Theorem 6.17]. We skip the details.

Theorem 5.30. *Suppose that $\nu \in \mathcal{C}_{r,n}$. Then $\{\cdot, \cdot\}_\nu$ defines a non-degenerate associative bilinear form on E^ν ; that is, E^ν is self-dual.*

Using this theorem and Corollary 5.27 we can easily get the next result.

Corollary 5.31. *Let $\lambda \in \mathcal{P}_{r,n}$. Then we have*

(i) *E^λ is a tilting module. Moreover, $E^\lambda \cong T^{\lambda'} \oplus \bigoplus_{\lambda' \succeq \mu} e_{\lambda\mu} T^\mu$ for some non-negative integers $e_{\lambda\mu}$.*

(ii) *T^λ is self-dual. Moreover, the tilting modules of YS_n^r are the indecomposable direct summands of the modules $\{E^\lambda \mid \lambda \in \mathcal{P}_{r,n}\}$.*

Recall that the Schur functor $F_n^r : \text{YS}_n^r\text{-mod} \rightarrow Y_{r,n}\text{-mod}$ defined in Proposition 4.3.

Lemma 5.32. *Suppose that $\mu \in \mathcal{C}_{r,n}$. Then $F_n^r(E^\mu) \cong N^\mu$ as $Y_{r,n}$ -modules.*

Proof. By Lemma 4.2 and Proposition 4.3 we have

$$F_n^r(E^\mu) = F_n^r(\dot{F}_n^\omega(\theta_\mu \dot{\text{YS}}_n^r)) = \dot{F}_n^r(\theta_\mu \dot{\text{YS}}_n^r) = \theta_\mu \dot{\text{YS}}_n^r \varphi_\omega \cong \text{Hom}_{Y_{r,n}}(Y_{r,n}, N^\mu) \cong N^\mu$$

as required. \square

Let $\mu, \nu \in \mathcal{C}_{r,n}$. Recall that for each $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$ there is a $Y_{r,n}$ -module homomorphism $\varphi'_{ST} : N^\nu \rightarrow N^\mu$; this induces a YS_n^r -module homomorphism $\Phi'_{ST} : E^\nu \rightarrow E^\mu$ given by $\Phi'_{ST}(\theta) = \varphi'_{ST}\theta$ for $\theta \in E^\nu$. The next proposition, which can be proved in exactly the same way as in [Ma2, Proposition 7.1], shows that these maps $\{\Phi'_{ST}\}$ give a basis of all the YS_n^r -module homomorphisms from E^ν to E^μ .

Proposition 5.33. *Suppose that $\mu, \nu \in \mathcal{C}_{r,n}$. Then $\text{Hom}_{\text{YS}_n^r}(E^\nu, E^\mu)$ is free as an \mathcal{R} -module with basis*

$$\{\Phi'_{ST} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } T \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \lambda \in \mathcal{P}_{r,n}\}.$$

By definition $E_n^r = \bigoplus_{\mu \in \mathcal{C}_{r,n}} E^\mu$ is a full tilting module for YS_n^r . Define the Ringel dual of YS_n^r to be the algebra $\text{End}_{\text{YS}_n^r}(E_n^r)$. If A is an algebra, let A^{op} be the opposite algebra in which the order of multiplication is reserved. The following corollary give a description of the Ringel dual of YS_n^r .

Corollary 5.34. *There exist canonical isomorphisms of \mathcal{R} -algebras $\text{End}_{\text{YS}_n^r}(E_n^r) \cong (\text{YS}_n^r)^{\text{op}}$.*

Corollary 5.35. *Suppose that $\lambda \in \mathcal{P}_{r,n}$. Then $F_n^r(T^{\lambda'}) \cong Y_\lambda$ as $Y_{r,n}$ -modules.*

Proof. By Lemma 5.32 the natural map $\text{Hom}_{Y_{r,n}}(N^\nu, N^\mu) \rightarrow \text{Hom}_{\text{YS}_n^r}(E^\nu, E^\mu)$ is injective; by Proposition 5.9(i) and Proposition 5.33 this is an isomorphism. Consequently, if an indecomposable tilting module T^ν is a direct summand of E^λ then $F_n^r(T^\nu)$ is an indecomposable direct summand of N^λ . Now, $E^\lambda \cong T^{\lambda'} \oplus \bigoplus_{\lambda' \succeq \mu} e_{\lambda\mu} T^\mu$ by Corollary 5.31(i) and $F_n^r(E^\lambda) \cong N^\lambda \cong Y_\lambda \oplus \bigoplus_{\nu \triangleright \lambda} c_{\nu\lambda} Y_\nu$ by Proposition 5.10(iii). Hence, the result follows by induction on the dominance order. \square

Corollary 5.36. *Let $\lambda, \mu \in \mathcal{P}_{r,n}$. Then $[T^\lambda : (W^\mu)^*] = [W^{\mu'} : L^{\lambda'}]$.*

Proof. Recall that $F_n^r(W^\lambda) \cong S^\lambda$ by Proposition 4.3. Since the functor F_n^r is easily seen to commute with duality, we have $F_n^r((W^\mu)^\otimes) \cong (F_n^r(W^\mu))^\otimes \cong S_{\mu'}$ by Corollary 5.16. Thus we have

$$\begin{aligned} [T^\lambda : (W^\mu)^\otimes] &= [F_n^r(T^\lambda) : F_n^r((W^\mu)^\otimes)] = [Y_{\lambda'} : S_{\mu'}] \\ &= [W_{\mu'} : L_{\lambda'}] = [W^{\mu'} : L^{\lambda'}], \end{aligned}$$

where the last equality follows from Proposition 5.9(iii). \square

6 Appendix. Cyclotomic Yokonuma-Schur algebras

In this appendix, we will generalize these results above to define and study the cyclotomic Yokonuma-Schur algebra by using the cellular basis of $Y_{r,n}^d(q)$ constructed in [C1]. Since this approach is very similar, we only mention the main results and skip all the details of the proofs.

We first recall the definition of the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$ and the construction of a cellular basis of it.

The affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n} = \widehat{Y}_{r,n}(q)$ is an \mathcal{R} -associative algebra generated by the elements $t_1, \dots, t_n, g_1, \dots, g_{n-1}, X_1^{\pm 1}$, in which the generators $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ satisfy the same relations as defined in $Y_{r,n}$, together with the following relations concerning the generators $X_1^{\pm 1}$:

$$\begin{aligned} X_1 X_1^{-1} &= X_1^{-1} X_1 = 1, \\ g_1 X_1 g_1 X_1 &= X_1 g_1 X_1 g_1, \\ X_1 g_i &= g_i X_1 \quad \text{for all } i = 2, 3, \dots, n-1, \\ X_1 t_j &= t_j X_1 \quad \text{for all } j = 1, 2, \dots, n. \end{aligned}$$

We define inductively elements X_2, \dots, X_n in $\widehat{Y}_{r,n}$ by

$$X_{i+1} := g_i X_i g_i \quad \text{for } i = 1, 2, \dots, n-1.$$

Then it is proved in [ChP1, Lemma 1] that we have, for any $1 \leq i \leq n-1$,

$$g_i X_j = X_j g_i \quad \text{for } j = 1, 2, \dots, n \text{ such that } j \neq i, i+1.$$

Let $d \geq 1$, and let $f_1 = (X_1 - v_1) \cdots (X_1 - v_d)$, where v_1, v_2, \dots, v_d are invertible indeterminates. Let \mathcal{J}_d denote the two-sided ideal of $\widehat{Y}_{r,n}$ generated by f_1 , and define the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d = Y_{r,n}^d(q)$ to be the quotient

$$Y_{r,n}^d = \widehat{Y}_{r,n} / \mathcal{J}_d.$$

It has been proved in [C1] (see also [ChP2, Theorem 4.15]) that the set of the following elements

$$\{X^\alpha t^\beta g_w \mid \alpha \in \mathbb{Z}_+^n \text{ with } \alpha_1, \dots, \alpha_n < d, \beta \in \mathbb{Z}_+^n \text{ with } \beta_1, \dots, \beta_n < r, w \in \mathfrak{S}_n\}$$

forms an \mathcal{R} -basis for $Y_{r,n}^d$.

Following [ChP2, Section 3.1], the combinatorial objects appearing in the representation theory of the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d$ will be m -compositions (resp. m -partitions) with $m = rd$, which can also be regarded as r -tuples of d -compositions (resp. d -partitions). We will call such an object an (r, d) -composition (resp. (r, d) -partition). By definition, an (r, d) -composition (resp. (r, d) -partition) of n is an ordered rd -tuple $\lambda = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)}))$ of compositions (resp. partitions) $\lambda_j^{(k)}$ such that $\sum_{k=1}^r \sum_{j=1}^d |\lambda_j^{(k)}| = n$. We denote by $\mathcal{C}_{r,n}^d$ (resp. $\mathcal{P}_{r,n}^d$) the set of (r, d) -compositions (resp. (r, d) -partitions) of n . We will say that the l -th composition (resp. partition) of the k -th r -tuple has position (k, l) .

Let $\lambda = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)}))$ be an (r, d) -composition of n . An (r, d) -tableau $\mathbf{t} = ((\mathbf{t}_1^{(1)}, \dots, \mathbf{t}_d^{(1)}), \dots, (\mathbf{t}_1^{(r)}, \dots, \mathbf{t}_d^{(r)}))$ of shape λ is obtained by replacing each node of λ by one of the integers $1, 2, \dots, n$, allowing no repeats. We will call the $\mathbf{t}_l^{(k)}$ the components of \mathbf{t} . Each node of \mathbf{t} is labelled by (a, b, k, l) if it lies in row a and column b of the component $\mathbf{t}_l^{(k)}$ of \mathbf{t} . An (r, d) -tableau \mathbf{t} of shape λ is row standard if each of its components is that. Similarly, \mathbf{t} is standard if each of its components is that and λ is an (r, d) -partition. The set of all standard (r, d) -partitions of shape λ is denoted by $\text{Std}(\lambda)$. We can define the (r, d) -tableau \mathbf{t}^λ of shape λ similarly.

For each (r, d) -composition λ of n , we have a Young subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1^{(1)}} \times \dots \times \mathfrak{S}_{\lambda_d^{(1)}} \times \dots \times \mathfrak{S}_{\lambda_1^{(r)}} \times \dots \times \mathfrak{S}_{\lambda_d^{(r)}},$$

which is exactly the row stabilizer of \mathbf{t}^λ .

For a row standard (r, d) -tableau \mathbf{s} of shape λ , let $d(\mathbf{s})$ be the element of \mathfrak{S}_n such that $\mathbf{s} = \mathbf{t}^\lambda d(\mathbf{s})$. Then $d(\mathbf{s})$ is a distinguished right coset representative of \mathfrak{S}_λ in \mathfrak{S}_n , that is, $l(wd(\mathbf{s})) = l(w) + l(d(\mathbf{s}))$ for any $w \in \mathfrak{S}_\lambda$. In this way, we obtain a correspondence between the set of row standard (r, d) -tableaux of shape λ and the set of distinguished right coset representatives of \mathfrak{S}_λ in \mathfrak{S}_n .

Definition 6.1. Let $\lambda = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathcal{C}_{r,n}^d$. Suppose that we choose all $1 \leq i_1 < i_2 < \dots < i_p \leq r$ such that $\lambda_{j_1}^{(i_1)}, \lambda_{j_2}^{(i_2)}, \dots, \lambda_{j_p}^{(i_p)}$ are the nonempty components of λ for some $1 \leq j_1, j_2, \dots, j_p \leq d$. Define $a_k := \sum_{j=1}^k |\lambda^{(i_j)}|$ for $1 \leq k \leq p$, where $|\lambda^{(i_j)}| = \sum_{l=1}^d |\lambda_l^{(i_j)}|$. Then the set partition A_λ associated with λ is defined as

$$A_\lambda := \{\{1, \dots, a_1\}, \{a_1 + 1, \dots, a_2\}, \dots, \{a_{p-1} + 1, \dots, n\}\},$$

which may be written as $A_\lambda = \{I_1, I_2, \dots, I_p\}$, and be referred to the blocks of A_λ in the order given above.

Definition 6.2. Let $\lambda = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathcal{C}_{r,n}^d$, and let $a_k := \sum_{j=1}^k |\lambda^{(i_j)}|$ ($1 \leq k \leq p$) be as above. Then we define

$$u_\lambda := u_{a_1, i_1} u_{a_2, i_2} \dots u_{a_p, i_p}.$$

Definition 6.3. Let $\lambda = ((\lambda_1^{(1)}, \dots, \lambda_d^{(1)}), \dots, (\lambda_1^{(r)}, \dots, \lambda_d^{(r)})) \in \mathcal{C}_{r,n}^d$. Associated with λ we can define the following elements a_l^k and b_k :

$$a_l^k = \sum_{m=1}^{l-1} |\lambda_m^{(k)}|, \quad b_k = \sum_{j=1}^{k-1} \sum_{i=1}^d |\lambda_i^{(j)}| \quad \text{for } 1 \leq k \leq r \text{ and } 1 \leq l \leq d.$$

Associated with these elements we can define an element $u_{\mathbf{a}}^+ = u_{\mathbf{a},1} u_{\mathbf{a},2} \cdots u_{\mathbf{a},r}$, where

$$u_{\mathbf{a},k} = \prod_{l=1}^d \prod_{j=1}^{a_l^k} (X_{b_k+j} - v_l).$$

Definition 6.4. Let $\lambda \in \mathcal{C}_{r,n}^d$ and define $u_{\mathbf{a}}^+$ as above. We set $U_{\lambda} := u_{\lambda} E_{A_{\lambda}}$, and define $x_{\lambda} = \sum_{w \in \mathfrak{S}_{\lambda}} q^{l(w)} g_w$. Then we define the element m_{λ} of $Y_{r,n}^d$ as follows:

$$m_{\lambda} := U_{\lambda} u_{\mathbf{a}}^+ x_{\lambda} = u_{\lambda} E_{A_{\lambda}} u_{\mathbf{a}}^+ x_{\lambda}.$$

Let $*$ denote the \mathcal{R} -linear anti-automorphism of $Y_{r,n}^d$, which is determined by

$$g_i^* = g_i, \quad t_j^* = t_j, \quad X_j^* = X_j \quad \text{for } 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq n.$$

Definition 6.5. Let $\lambda \in \mathcal{C}_{r,n}^d$, and let \mathfrak{s} and \mathfrak{t} be row standard (r, d) -tableaux of shape λ . We then define $m_{\mathfrak{s}\mathfrak{t}} = g_{d(\mathfrak{s})}^* m_{\lambda} g_{d(\mathfrak{t})}$.

For each $\mu \in \mathcal{P}_{r,n}$, let $Y_{r,n}^{d, \triangleright \mu}$ be the \mathcal{R} -submodule of $Y_{r,n}^d$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in \text{Std}(\lambda)$ for various $\lambda \in \mathcal{P}_{r,n}$ such that $\lambda \triangleright \mu$.

Theorem 6.6. (See [C1, Theorem 6.18].) *The algebra $Y_{r,n}^d$ is a free \mathcal{R} -module with a cellular basis*

$$\mathcal{B}_{r,n}^d = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } (r, d)\text{-partition } \lambda \text{ of } n\},$$

that is, the following properties hold.

(i) The \mathcal{R} -linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$ ($m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{B}_{r,n}^d$) is an anti-automorphism on $Y_{r,n}$.

(ii) For a given $h \in Y_{r,n}^d$ and $\mathfrak{t} \in \text{Std}(\mu)$, there exist $r_{\mathfrak{v}} \in \mathcal{R}$ such that for all $\mathfrak{s} \in \text{Std}(\mu)$, we have

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \text{Std}(\mu)} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \quad \text{mod } Y_{r,n}^{d, \triangleright \mu},$$

where $r_{\mathfrak{v}}$ may depend on $\mathfrak{v}, \mathfrak{t}$ and h , but not on \mathfrak{s} .

For $\lambda \in \mathcal{C}_{r,n}^d$, a λ -tableau $S = ((S_1^{(1)}, \dots, S_d^{(1)}), \dots, (S_1^{(r)}, \dots, S_d^{(r)}))$ is a map $S : [\lambda] \rightarrow \{1, \dots, n\} \times \{1, \dots, d\} \times \{1, \dots, r\}$, which can be regarded as the diagram $[\lambda]$, together with an ordered triple (i, j, k) ($1 \leq i \leq n, 1 \leq j \leq d, 1 \leq k \leq r$) attached to each node. Given $\lambda \in \mathcal{P}_{r,n}^d$ and $\mu \in \mathcal{C}_{r,n}^d$, a λ -tableau S is said to be of type μ if the number of (i, j, k) in the entry of S is equal to $\mu_{j,i}^{(k)}$. Given $\mathfrak{s} \in \text{Std}(\lambda)$, $\mu(\mathfrak{s})$, a

λ -tableau of type μ , is defined by replacing each entry m in \mathfrak{s} by (i, j, k) if m is in the i -th row of the (k, j) -th component of \mathfrak{t}^μ .

We define a total order on the set of triples (i, j, k) by $(i_1, j_1, k_1) < (i_2, j_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $j_1 < j_2$, or $k_1 = k_2$, $j_1 = j_2$, and $i_1 < i_2$. Let $\lambda \in \mathcal{P}_{r,n}^d$ and $\mu \in \mathcal{C}_{r,n}^d$. Suppose that $S = ((S_1^{(1)}, \dots, S_d^{(1)}), \dots, (S_1^{(r)}, \dots, S_d^{(r)}))$ is a λ -tableau of type μ . S is said to be semistandard if each component $S_j^{(k)}$ is non-decreasing in rows, strictly increasing in columns, and all entries of $S_j^{(k)}$ are of the form (i, h, l) with $h \geq j$ and $l \geq k$. We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard λ -tableaux of type μ .

For any $\kappa \in \mathcal{C}_{r,n}^d$, we define its type $\alpha(\kappa)$ by $\alpha(\kappa) = (n_1, \dots, n_r)$ with $n_i = |\kappa^{(i)}|$. Assume that $\lambda \in \mathcal{P}_{r,n}^d$ and $\mu \in \mathcal{C}_{r,n}^d$. We define a subset $\mathcal{T}_0^+(\lambda, \mu)$ of $\mathcal{T}_0(\lambda, \mu)$ by

$$\mathcal{T}_0^+(\lambda, \mu) = \{S \in \mathcal{T}_0(\lambda, \mu) \mid \alpha(\lambda) = \alpha(\mu)\}.$$

For each $\mu \in \mathcal{C}_{r,n}^d$, let $M^\mu = m_\mu Y_{r,n}^d$. We now construct a basis of M^μ related to the cellular basis $\{m_{\mathfrak{st}}\}$ in Theorem 6.6. For $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$, we define

$$m_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{st}}.$$

The following theorem can be proved in exactly the same way as in [DJM, Theorem 4.14] by combining [DJM] and [C1].

Theorem 6.7. *Let $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $\mathfrak{t} \in \text{Std}(\lambda)$ for some $\lambda \in \mathcal{P}_{r,n}^d$ and $\mu \in \mathcal{C}_{r,n}^d$. Then $m_{S\mathfrak{t}} \in M^\mu$. Moreover, M^μ is free with an \mathcal{R} -basis*

$$\{m_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } \mathfrak{t} \in \text{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}^d\}.$$

Let $\mu, \nu \in \mathcal{C}_{r,n}^d$ and $\lambda \in \mathcal{P}_{r,n}^d$. We assume that $\alpha(\mu) = \alpha(\nu) = \alpha(\lambda)$. For $S \in \mathcal{T}_0^+(\lambda, \mu)$, $T \in \mathcal{T}_0^+(\lambda, \nu)$, put

$$m_{ST} = \sum_{\mathfrak{s}, \mathfrak{t}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{st}},$$

where the sum is taken over all $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$ and $\nu(\mathfrak{t}) = T$. We then have the next proposition by making use of Theorem 6.7.

Proposition 6.8. *Suppose that $\mu, \nu \in \mathcal{C}_{r,n}^d$ with $\alpha(\mu) = \alpha(\nu)$. Then the set*

$$\{m_{ST} \mid S \in \mathcal{T}_0^+(\lambda, \mu) \text{ and } T \in \mathcal{T}_0^+(\lambda, \nu) \text{ for some } \lambda \in \mathcal{P}_{r,n}^d\}$$

is an \mathcal{R} -basis of $M^{\nu} \cap M^\mu$.*

Definition 6.9. Suppose that $M_n^{r,d} = \bigoplus_{\mu \in \mathcal{C}_{r,n}^d} M^\mu$. We define the cyclotomic Yokonuma-Schur algebra $\text{YS}_n^{r,d}$ as the endomorphism algebra

$$\text{YS}_n^{r,d} = \text{End}_{Y_{r,n}^d}(M_n^{r,d}),$$

which is isomorphic to $\bigoplus_{\mu, \nu \in \mathcal{C}_{r,n}^d} \text{Hom}_{Y_{r,n}^d}(M^\nu, M^\mu)$.

Let $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$. In view of Proposition 6.8, we can define $\varphi_{ST} \in \text{Hom}_{Y_{r,n}^d}(M^\nu, M^\mu)$ by

$$\varphi_{ST}(m_\nu h) = m_{ST} h$$

for all $h \in Y_{r,n}^d$. We extend φ_{ST} to an element of $\text{YS}_n^{r,d}$ by defining φ_{ST} to be zero on M^κ for any $\nu \neq \kappa \in \mathcal{C}_{r,n}^d$. For each $\lambda \in \mathcal{P}_{r,n}^d$, let $\mathcal{T}_0^+(\lambda) = \bigcup_{\mu \in \mathcal{C}_{r,n}^d} \mathcal{T}_0^+(\lambda, \mu)$. We denote by $\text{YS}_{r,n}^{d, \triangleright \lambda}$ the \mathcal{R} -submodule of $\text{YS}_n^{r,d}$ spanned by φ_{ST} such that $S, T \in \mathcal{T}_0^+(\alpha)$ with $\alpha \triangleright \lambda$. Then we can prove the following theorem by a similar argument as in [DJM, Theorem 6.6].

Theorem 6.10. *The Yokonuma-Schur algebra $\text{YS}_n^{r,d}$ is free as an \mathcal{R} -module with a basis*

$$\{\varphi_{ST} \mid S, T \in \mathcal{T}_0^+(\lambda) \text{ for some } \lambda \in \mathcal{P}_{r,n}^d\}.$$

Moreover, this basis satisfies the following properties.

- (i) The \mathcal{R} -linear map $*$: $\text{YS}_n^{r,d} \rightarrow \text{YS}_n^{r,d}$ determined by $\varphi_{ST}^* = \varphi_{TS}$, for all $S, T \in \mathcal{T}_0^+(\lambda)$ and all $\lambda \in \mathcal{P}_{r,n}^d$, is an anti-automorphism of $\text{YS}_n^{r,d}$.
- (ii) Let $T \in \mathcal{T}_0^+(\lambda)$ and $\varphi \in \text{YS}_n^{r,d}$. Then for each $V \in \mathcal{T}_0^+(\lambda)$, there exists $r_V = r_{V,T,\varphi} \in \mathcal{R}$ such that for all $S \in \mathcal{T}_0^+(\lambda)$, we have

$$\varphi_{ST}\varphi \equiv \sum_{V \in \mathcal{T}_0^+(\lambda)} r_V \varphi_{SV} \pmod{\text{YS}_{r,n}^{d, \triangleright \lambda}}.$$

In particular, this basis $\{\varphi_{ST}\}$ is a cellular basis of $\text{YS}_n^{r,d}$.

Now we can apply the general theory of cellular algebras in view of Theorem 6.10. For example, we can easily give a complete set of non-isomorphic irreducible $\text{YS}_n^{r,d}$ -modules over an arbitrary field, and further prove that $\text{YS}_n^{r,d}$ is a quasi-hereditary algebra. For the cyclotomic Yokonuma-Schur algebra $\text{YS}_n^{r,d}$, we can also define the Schur functor from the category of $\text{YS}_n^{r,d}$ -modules to the category of $Y_{r,n}^d$ -modules and the tilting modules for it in exactly the same way as in Sections 4 and 5, we skip all the details and leave them to the reader.

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